ON SPECIAL DIOPHANTINE TRIPLETS INVOLVING CENTERED SQUARE NUMBERS

ABSTRACT

The identification of Diophantine pairs or triples has so far included several researchers. Special Diophantine Triples have recently taken on a distinctive role and served as an extension of these Diophantine pairs and triples. Similarly, we seek to compile all possible Special Diophantine triples involving the well-known centered square numbers in this manuscript.

Keywords: *Diophantine m – Tuple, Pell Equation, Centered Square Number, Special Diophantine Triplet.*

Introduction:

The study of integers is the main focus of the field of pure mathematics known as number theory. It has long been a favoured topic among math students and instructors. It is an old subject with a large amount of information. It is regarded as the "purest" aspect of mathematics. Number theory is one of the most crucial fields of mathematics nowadays for applications in safe information sharing and cryptography.

For those who desire to master numbers, it contains a wide range of results from the simple to the complex. In Number Theory, integers are characterized as solutions to the Diophantine equation$(ax + by = c)$, which is a remarkable description. A Diophantine equation is a polynomial equation involving two or more unknowns with all constants being integers and only integer solutions are of interest. An example is $3x + 7y = 1$, where x, y and z are all integers.

The term Diophantine refers to Diophantus of Alexandria, a third century Greek mathematician who was one of the first to introduce symbols into variable-based mathematics. Hilbert's tenth issue is one of the more well-known Diophantine problems. It asks, given a Diophantine equation with any number of unknown quantities and rational integral coefficients,

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how can one determine in a finite number of operations if the equation can be solved in rational integers? Yuri Matiyasevich was the first to find a solution to this issue in 1970. Diophantus discovered a slew of new results and definitions during his life time, including the Diophantine m-tuple, Diophantine equation, and so on.

Diophantus started by investigating the problem of obtaining rational quadruples, citing one example: $\{\frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16}\}$ [6]. Fermat discovered the first integer quadruple, $\{1, 3, 8, 120\}$[3]. There are an infinite number of integer Diophantine quadruples known, and no integer Diophantine quintuple is thought to exist. Dujella [4] virtually confirmed this by demonstrating that there can be only a finite number of Diophantine quintuples and that all of them are effectively computable.

The occurrence of Diophantine triples and quadruples with the property $D(n)$ for any integer n and, in addition, for any linear polynomial in n has been studied by a number of mathematicians. In this unusual situation, a complete review of several Diophantine triplet issues can be found.

In this study, "Diophantine m-tuple with the property $D(n)$ refers to the set of m distinct integers $(a_1, a_2, \cdots, a_m)$ that satisfy the requirement that the product of any two different elements plus n is a perfect square, where n is a non-zero integer. When $m = 3$, the set of integers (a_1, a_2, a_3) with the property $D(1)$ meets the requirement that $a_i a_j + 1$ is a perfect square for all, $1 \leq i, j \leq 3$ *and* $i \neq j$, which is known as a Diophantine 3-tuple or Diophantine triplet"[3].

A centred square number is a centred figurate number that indicates the total number of dots in a square that have successive square layers with a dot in the centre and all other dots surrounding it. In other words, the sum of two successive square numbers is a centred square number. $(i.e., x^2 + (x-1)^2)$. The purpose of this paper is to construct Special Diophantine triplets involving centered square numbers of distinct ranks with different properties.

Section 2 contains the definition of special Diophantine triplets. The main work is discussed in Section 3. In subsection 3.1, we construct special Diophantine triplets involving centered square numbers of rank r and $r-1$ with the property $D(\alpha_1)$. Similarly subsection 3.2 discusses rank r and $r-2$ with the property $D(\alpha_2)$, and the following one discusses rank r and $r-3$ with the property $D(\alpha_3)$. We present some properties of centered square numbers in the final subsection.

2. Essential Notations and Definitions

Throughout this paper, "$C_{4,r}$" stands for Centered square number of rank r.

Definition 2.1. [12] A set of three different polynomials with integer coefficients (a_1, a_2, a_3) is said to be a special Diophantine Triplets with the property $D(n)$ if $a_i a_j + a_i + a_j + n$ is a perfect square for all $1 \leq i, j \leq 3$ *and* $i \neq j$, where n may be non – zero polynomial with integer coefficients.

3. Analytical Technique

3.1 Generating the incredible Special Diophantine Triplets with the characteristic $D(\alpha_1)$ using centered square numbers of rank r *and* $r-1$.

Let μ_r and μ_{r-1} be centered square numbers of rank r and $r-1$ respectively.Then $\mu_r = 2r^2 - 2r + 1$ and $\mu_{r-1} = 2r^2 - 6r + 5$. Let $D(\alpha_1) = -3r^4 + 6r^3 + 3r^2 - 6r - 2$.

Now, $\mu_r\mu_{r-1} + \mu_r + \mu_{r-1} + D(\alpha_1) = r^4 - 10r^3 + 31r^2 - 30r + 9 = (r^2 - 5r + 3)^2 = \beta_1^2$, where, $\beta_1 = r^2 - 5r + 3$.

Let v be a non-zero integer such that

$\mu_r v + \mu_r + v + D(\alpha_1) = \beta_2^2$ (1)

$\mu_{r-1} v + \mu_{r-1} + v + D(\alpha_1) = \beta_3^2$ (2)

Eliminating v between (1) and (2) by letting $\beta_2 = x + (\mu_r + 1)y$ and $\beta_3 = x + (\mu_{r-1} + 1)y$, we get $x^2 - (4r^4 - 16r^3 + 28r^2 - 24r + 12)y^2 = D(\alpha_1) - 1$. We have, $v = 2x_0y_0 + y_0^2(\mu_r + \mu_{r-1} + 2) - 1$. Taking $x_0$and y_o,

$v = 2(-r^2 + 5r - 3) + \mu_r + \mu_{r-1} + 1$

$= -2r^2 + 10r - 6 + 2r^2 - 2r + 1 + 2r^2 - 6r + 5 + 1 = 2r^2 + 2r + 1$. That is $v = 2r^2 + 2r + 1$. Therefore, $v = C_{4,r+1,}$ which is a centered square number of rank $r + 1$.

The triplets $(\mu_r, \mu_{r-1}, v) = (2r^2 - 2r + 1, 2r^2 - 6r + 5, 2r^2 + 2r + 1)$ is a special Diophantine Triplets with the attribute $D(\alpha_1)$ is mentioned in the table for $r = 1, 2, \ldots, 6$.

$\boldsymbol{n}$	**Special Diophantine Triplet**	$D(\alpha_1) = -3r^4 + 6r^3 + 3r^2 - 6r - 2$
1	(1,1,5)	-2
2	(5,1,13)	-2
3	$(13, 5, 25)$	-74
4	$(25, 13, 41)$	-362
5	$(41, 25, 61)$	-1082
6	$(61, 41, 85)$	-2522

3.2. Generating the incredible Special Diophantine Triplets with the characteristic $D(\boldsymbol{\alpha_2})$ using centered square numbers of rank $\boldsymbol{r\ and\ r-2}$.

Let μ_r and μ_{r-2} be centered square numbers of rank r and $r-2$ respectively.Then $\mu_r = 2r^2 - 2r + 1$ and $\mu_{r-1} = 2r^2 - 10r + 13$. Let $D(\alpha_2) = -3r^4 + 6r^3 + 39r^2 - 42r - 2$.

Now, $\mu_r\mu_{r-2} + \mu_r + \mu_{r-2} + D(\alpha_2) = r^4 - 18r^3 + 91r^2 - 90r + 25 = (r^2 - 9r + 5)^2 = \gamma_1^2$, where, $\gamma_1 = r^2 - 9r + 5$.

Let η be a non-zero integer such that

$\mu_r\eta + \mu_r + \eta + D(\alpha_2) = \gamma_2^2$ (3)

$\mu_{r-2}\eta + \mu_{r-2} + \eta + D(\alpha_2) = \gamma_3^2$ (4)

Eliminating η between (3) and (4) by setting $\gamma_2 = x + (\mu_r + 1)y$ and $\gamma_3 = x + (\mu_{r-2} + 1)y$, we get $x^2 - (4r^4 - 24r^3 + 52r^2 - 48r + 28)y^2 = D(\alpha_2) - 1$ with initial conditions $y_0 = 1\ and\ x_0 = -r^2 + 9r - 5$. We have, $c = 2x_0y_0 + y_0^2\,(\mu_r + \mu_{r-2} + 2) - 1$. puting $x_0$and y_o,

$\eta = 2(-r^2 + 9r - 5) + \mu_r + \mu_{r-2} + 1$

$= -2r^2 + 18r - 10 + 4r^2 - 12r + 13 = 2r^2 + 6r + 5$. That is $\eta = 2r^2 + 6r + 5$. Therefore, $v = C_{4,r+2,}$ which is a centered square number of rank $r + 2$.

The triplets $(\mu_r, \mu_{r-2}, \eta) = (2r^2 - 2r + 1, 2r^2 - 10r + 13, 2r^2 + 6r + 5)$ is a special Diophantine Triplets with the property $D(-3r^4 + 6r^3 + 39r^2 - 42r - 2$. The following table list out the Special Diophantine Triplet with the property $D(\alpha_2\ )$ for $r = 1, 2, \ldots, 6$.

n	**Special Diophantine Triplet**	$D(\alpha_1) = -3r^4 + 6r^3 + 39r^2 - 42r - 2$
1	(1,5,13)	-2
2	(5,1,25)	70
3	(13, 1, 41)	142
4	(25, 5, 61)	70
5	(41, 13,25)	-362
6	(61, 25, 41)	-1442

3.3. Generating the incredible Special Diophantine Triplets with the characteristic $D(\alpha_3)$ using centered square numbers of rank r *and* $r-3$.

Let $\mu_r = C_{4,r}$ and $\mu_{r-3} = C_{4,r-3}$ be centered square numbers of rank r and $r-3$ respectively.Then $\mu_r = 2r^2 - 2r + 1$ and $\mu_{r-3} = 2r^2 - 14r + 25$. Let $D(\alpha_3) = -3r^4 + 6r^3 + 99r^2 - 102r - 2$.

Now, $\mu_r\mu_{r-3} + \mu_r + \mu_{r-3} + D(\alpha_3) = r^4 - 26r^3 + 155r^2 - 182r + 49 = (r^2 - 13r + 7)^2 = \delta_1^2$, where, $\delta_1 = r^2 - 13r + 7$.

Let ζ be a non-zero integer such that
$\mu_r\zeta + \mu_r + \zeta + D(\alpha_3) = \delta_2^2$ (5)
$\mu_{r-3}\zeta + \mu_{r-3} + \zeta + D(\alpha_3) = \delta_3^2$

ζ is removed from (5) and (6) by letting $\delta_2 = x + (\mu_r + 1)y$ and $\gamma_3 = x + (\mu_{r-3} + 1)y$, we get, $x^2 - (4r^4 - 32r^3 + 84r^2 - 80r + 52)y^2 = D(\alpha_3) - 1$ with initial conditions $y_0 = 1$ *and* $x_0 = -r^2 + 13r - 7$. We have, $\zeta = 2x_0y_0 + y_0^2(\mu_r + \mu_{r-3} + 2) - 1$. puting $x_0$and y_o,

$\zeta = 2(-r^2 + 5r - 3) + \mu_r + \mu_{r-3} + 1$

$= -2r^2 + 26r - 14 + 2r^2 - 2r + 1 + 2r^2 - 14r + 25 + 1 = 2r^2 + 10r + 13$. That is $\zeta = 2r^2 + 10r + 13$. Therefore, $\zeta = C_{4,r+3}$, which is a centered square number of rank $r + 3$.

The triplets $(\mu_r, \mu_{r-3}, \zeta) = (2r^2 - 2r + 1, 2r^2 - 14r + 25, 2r^2 + 10r + 13)$ is a special Diophantine Triplets with the property $D(-3r^4 + 6r^3 + 99r^2 - 102r - 2$. The following table list out the Special Diophantine Triplet with the property $D(\alpha_3$) for $r = 1, 2, \ldots, 6$.

n	**Special Diophantine Triplet**	$D(\alpha_1) = -3r^4 + 6r^3 + 39r^2 - 42r - 2$
1	(1,13,25)	-2
2	(5,5,41)	190
3	(13,1,61)	502
4	(25, 1,85)	790
5	(41, 5,113)	838
6	(61,13,145)	358

3.4. Some Peculiar Properties of Centered Square Numbers.

1. The one's digit of each of the Centered Square Numbers follows the pattern, and they are all odd. 1 – 5 – 3 – 5 – 1 - . . .

For, $C_{4,r} = r^2 + (r-1)^2$.
$= 2r^2 - 2r + 1$. This shows that $C_{4,r}$ is a odd number.

2. All of the Centered Square Numbers have the $4k + 1$ formula, indicating that they are all Hilbert Numbers. However, the reverse need not be true.

3. Every Centered Square Number can be written as the sumof two adjacent square numbers, and it is not necessary for the converse to be true.

4. Every number r in this triplet is half of the r^{th} odd square number plus 1.

For, $r = \frac{(2r^2+1)^2+1}{2} = \frac{4r^2+4r+2}{2}$

$= \frac{2(2r^2+2r+1)}{2} = 2r^2 + 2r + 1$, which is a centered square number.

5. Every number r in this triplet also can be expressed in terms of Triangular numbers and Centered triangular numbers.

For, $r = 1 + 4T_{r-1}$, $T_r = \frac{r(r+1)}{2}$ is a triangular number. $n = 1 + 4\frac{(r-1)(r+1-1)}{2} = 1 + 2(r^2 - 1) = \ =$ $2r^2 - 2r + 1$, which is a centered square number.

6. Every number r in this triplet also can be expressed in terms of Centered Triangular numbers.

For, $r = \frac{4C_{3,r-1}}{3}$, where $C_{3,r} = 1 + 3(\frac{r(r-1)}{2})$ is a centered square number.

$n = \frac{4\left[1+3\left(\frac{r(r-1)}{2}\right)\right]-1}{3} \ = \frac{6r^2-6r+3}{3} = 2r^2 - 2r + 1$, which is a centered square number.

7. Every Centered Square number except 1 is the hypotenuse of a Pythagorean Triplet. For Example, $(3, 4, 5), (5,12, 13)$. This is exactly the sequence of Pythagorean Triples where the two longest sides differ by 1.

8. each of the three numbers in this triplet is the product of two octahedral integer's thet follow each othe.

For, $r = \ O_r - O_{r-1}$, where $O_r = \frac{r(2r^2+1)}{3}$ is an octahedral number.

$r = O_r - O_{r-1} = \frac{r(2r^2+1)}{3} - \frac{(r-1)(2(r-1)^2+1)}{3} = \frac{6r^2-6r+3}{3} = 2r^2 - 2r + 1$, which is a centered square number.

4. Conclusion

This paper has been demonstrated how to build the Special Diophantine Triplets with the property $D(n)$ using centered square numbers. This results may be expanded to Quadruples and Quintuples.

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