SOME ANNOTATIONS ON ALMOST AND PSEUDO ALMOST EQUILATERAL RATIONAL RECTANGLES

ABSTRACT

A rational rectangle is defined as a rectangle with rational sides and rational diagonal. Using this here we define new collection of rectangles namely Almost Equilateral Rational Rectangle and Pseudo Almost Equilateral Rational Rectangle. Rectangles with sides $n, n \pm 1$ *where* n *is a rational number come under the name Almost Equilateral Rational Rectangles. Extending this, we define pseudo almost equilateral rational rectangles as rectangles with sides* n *and* $n \pm r$ *where* $n \in \mathbb{Q}$ *and* r *is a positive integer greater than one. In this paper, we aim to collect all such rectangles with rational diagonal and area equal to the perimeter.*

Keywords: *Rational rectangle, almost and pseudo almost equilateral rational rectangle.*

MSC2020 Mathematics Subject Classifications: *11A55, 11B37, 11D09, 11Y65.*

1. Introduction:

Number theory is one of the classical branches of mathematics. It has been a treasured subject for all mathematicians who wish to handle numbers incredibly. This theory deals with counting numbers and later discusses their analytic properties also [8]. Apart from these, there are so many valuable things found in Number theory. One such thing is Diophantine Analysis. This part deals with Diophantine equations and their solutions [3]. The beauty of this part is finding the solutions because there is no common method found to solve any Diophantine equation.

In recent days, one of the attractive research ideas is solving the Diophantine equation. Even though there is no common method to solve them, each equation is solved uniquely. It is evident in [7]. The specialty of these types of equations is on one side the theory (finding solutions) is developing and on the other side, the researchers are using those solutions to solve problems in areas such as combinatorics. Likewise, an interesting thing is the solutions are used in geometry.

In [3], Dickson provided some evidence for geometry is connected with number theory. The initial idea behind such work is to collect a particular geometrical shape with certain conditions imposed on them. Later, existing regular shapes are given different names under some conditions and their properties are studied. [6] is a suitable example of this. Like this, we work on existing rational rectangles, rectangles with rational sides, and rational diagonals.

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An equilateral rectangle is defined to be a rectangle with all sides equal. That is a square. Combining this with the rational rectangle, we construct two new kinds of rectangles, namely Almost Equilateral Rational Rectangle and Pseudo Almost Equilateral Rational Rectangle. An Almost Equilateral Rational Rectangle is defined as a rectangle with sides n and $n \pm 1$ where n is a rational number, whereas a Pseudo Almost Equilateral Rational Rectangle is one with sides n and $n \pm r$ where n is rational and r is a positive integer greater than 1. For example, a rectangle with sides $\frac{3}{2}$ and $\frac{13}{2}$ is a pseudo almost equilateral rational rectangle.

In this paper, we collect our newly defined rectangles with two properties. One is a rational diagonal and the other one is the area that coincides with the perimeter. To do so, we make use of a special kind of Diophantine equation, namely the Pell and negative Pell equation. The solutions to such equations are dealt with in [1,7]. Also, the elementary Diophantine equation of degree 2 is used in this work.

Excluding the introduction and conclusion, this manuscript contains four sections. Section (2) discusses some preliminary results. Section (3) collects all almost equilateral rational rectangles with rational diagonal, whereas section (4) collects all pseudo almost equilateral rational rectangles with rational diagonal. Section (5) concentrates on an almost and pseudo almost equilateral rational rectangle in which the area coincides with the perimeter.

2. Preliminaries

In this section, we provide the solutions to the negative Pell equation $x^2 - 2y^2 = -1$ in terms of the solutions of the Pell equation $x^2 - 2y^2 = 1$. Also, we provide the solutions of $x^2 - 2y^2 = -r^2$.

Lemma 2.1. [7] Let u and v be integers satisfying $u^2 - 2v^2 = 1$ such that $\frac{u-1}{2}$ and $\frac{u+1}{4}$ are perfect squares. Then the integers x and y satisfy the equation $x^2 - 2y^2 = -1$ is of the form $x = \sqrt{\frac{u-1}{2}}$ and $y = \sqrt{\frac{u+1}{4}}$.

Proof. It is clear that u is odd and v is even. The relation $u^2 - 2v^2 = 1$ can be written as $u^2 - 1 = 2v^2$. The term $u^2 - 1$ is factored as $u^2 - 1 = (u+1)(u-1)$. Thus one can get $2v^2 = (u+1)(u-1)$. Since the gcd $(u-1, u+1) = 2$, one of the factors must be of form $2x^2$ and the other is of form $4y^2$. Suppose $u + 1 = 2x^2$ and $u - 1 = 4y^2$. Then it gives the relation $x^2 - 2y^2 = 1$. But $u - 1 = 2x^2$ and $u + 1 = 4y^2$ leads to the negative Pell equation $x^2 - 2y^2 = -1$.

Lemma 2.2. If u and v are integers such that $u^2 - 2v^2 = -1$, then $x = ur$ and $y = vr$ satisfies $x^2 - 2y^2 = -r^2$, where r is an integer.

Proof. It is obvious that, $x^2 - 2y^2 = u^2r^2 - 2v^2r^2 = -r^2$.

Remark 1. By lemmas (2.1) and (2.2), it is clear that $x = r\sqrt{\frac{u-1}{2}}$ and $y = r\sqrt{\frac{u+1}{4}}$ satisfies $x^2 - 2y^2 = -r^2$.

3. Almost Equilateral Rational Rectangles with Rational Diagonal

In this section, we aim to collect all almost equilateral rational rectangles with rational diagonals using the solutions of $x^2 - 2y^2 = \pm 1$. In each case, we obtain the generalized value of n**.**

Theorem 3.1. Let R be a rectangle with sides $n, n+1$ where $n \in \mathbb{Q}$. If the diagonal of R is an integer, then $n = \frac{1}{2}\left[\sqrt{\frac{x-1}{2}} - 1\right]$ where x is an integer such that $x^2 - 2y^2 = 1$ for some $y \in \mathbb{Z}$.

Proof. Let us prove the theorem by considering two cases. Suppose n is an integer and h is the diagonal of R. Then by Pythagoras theorem, we have $n^2 + (n+1)^2 = h^2$. This equation reduces to $k^2 - 2h^2 = -1$ where $k = 2n+1$. We have to find $n \in \mathbb{Z}$ such that $h \in \mathbb{Z}$. For that purpose, the equation $k^2 - 2h^2 = -1$ needs to be solved over $\mathbb{Z}$. From lemma (2.1), it is clear that $k = \sqrt{\frac{x-1}{2}}$ and $h = \sqrt{\frac{x+1}{4}}$ for some $x \in \mathbb{Z}$ such that $x^2 - 2y^2 = 1$ where $y \in \mathbb{Z}$. Now we claim that n obtained from this k is an integer. It is enough to show that k is odd. If k is even, then $k = 2k_1$ for some $k_1 \in \mathbb{Z}$. This gives $x = 8k_1^2 + 1$. But this value of x doesn't satisfy the basic nature of h. Thus k must be odd and so n is an integer. Also, n is of the form $n = \frac{k-1}{2} = \frac{1}{2}\left[\sqrt{\frac{x-1}{2}} - 1\right]$.

Let us move to the other case that $n \in \mathbb{Q}\backslash\mathbb{Z}$. If $n = \frac{r}{s} (\in \mathbb{Q}\backslash\mathbb{Z})$, then the equation from the Pythagoras theorem implies $h^2 = \frac{2r^2 + 2rs + s^2}{s^2}$. As h is an integer, h^2 is also an integer. This leads to the fact that $2r(r+s) = (t-1)s^2$ for some $t \in \mathbb{Z}$ and so $s^2 \mid 2r(r+s)$. Since gcd $(r,s) = 1$, we have gcd $(r, s^2) = 1$. Thus $s^2 | 2(r+s)$. In the usual sense, this can be written as $2(r+s) = t_1 s^2$ for some $t_1 \in \mathbb{Z}$. Rewriting this as $t_1 = \frac{2r}{s^2} + \frac{2}{s}$, we see that $t_1 \notin \mathbb{Z}$, a contradiction. Thus in this case we have no such required n. This completes the proof.

Theorem 3.2. Let R be a rectangle with sides $n, n-1$ where $n \in \mathbb{Q}$. If the diagonal of R is an integer, then $n = \frac{1}{2}\left[\sqrt{\frac{x-1}{2}} + 1\right]$ where x is an integer such that $x^2 - 2y^2 = 1$ for some $y \in \mathbb{Z}$.

Proof. The proof is the same as in theorem (3.1).

Theorem 3.3. Let R be a rectangle with sides $n, n+1$ where $n \in \mathbb{Q}$. If the diagonal of R is a rational number of the form $\frac{p}{q}, (p,q) = 1$, then $n = \frac{a^2-b^2}{b^2-a^2+2ab}$ for some $a, b \in \mathbb{Z}$.

Proof. Let h be the diagonal of R. In this theorem, we aim to collect $n \in \mathbb{Q}$ such that $h \in \mathbb{Q}\backslash\mathbb{Z}$. If $n \in \mathbb{Z}$, then $n^2 + (n+1)^2 = h^2 \in \mathbb{Z}$ and so $h \in \mathbb{Z}$. Thus no required h exists when $n \in \mathbb{Z}$. If $n = \frac{u}{v} \in \mathbb{Q}$, $n^2 + (n+1)^2 = h^2$ implies $u^2 + (u+v)^2 = \left(\frac{pv}{q}\right)^2$. Since $u, v \in \mathbb{Z}$, we must have $q|v$. Take $v = kq$ for some $k \in \mathbb{Z}$. Replacing this in the last expression, we get an equation of the form $x^2 + y^2 = z^2$ where $x = u, y = u + kq, z = pk$. This equation has to be solved over $\mathbb{Z}$. As this is a Pythagorean equation, its general solution is

$$x = \alpha\,(a^2 - b^2)$$
$$y = 2\,\alpha\, ab$$
$$z = \alpha(a^2 + b^2)$$

where $\alpha, a, b \in \mathbb{Z}$. That is, the solution is

$$u = \alpha\,(a^2 - b^2)$$

$$u + kq = 2\,\alpha\, ab$$

$$pk = \alpha(a^2 + b^2)$$

Considering all these, we obtain $v = \alpha\,(b^2 - a^2 + 2ab)$ and so $n = \frac{a^2-b^2}{b^2-a^2+2ab}$.

Theorem 3.4. Let R be a rectangle with sides $n, n-1$ where $n \in \mathbb{Q}$. If the diagonal of R is a rational number of the form $\frac{p}{q}, (p,q) = 1$, then $n = \frac{a^2-b^2}{a^2-b^2-2ab}$ for some $a, b \in \mathbb{Z}$.

Proof. Following the same steps as in theorem (3.3), we obtain

$$u = \alpha\,(a^2 - b^2)$$

$$u - kq = 2\,\alpha\,ab$$

$$pk = \alpha(a^2 + b^2)$$

This gives us $v = \alpha\,(a^2 - b^2 - 2ab)$. Hence $n = \frac{a^2-b^2}{a^2-b^2-2ab}$.

4. Pseudo Almost Equilateral Rational Rectangles with Rational Diagonal

As in section (3), this section collects all pseudo almost equilateral rational rectangles with rational diagonal. Here also general form of n is given which is obtained from the solutions of the equation $x^2 + y^2 = z^2$.

Theorem 4.1. Let R be a rectangle with sides $n, n+r$ where $n \in \mathbb{Q}$ and $r \in \mathbb{N}$ such that $r > 1$. If the diagonal of R is an integer, then $n = \frac{r}{2}\left[\sqrt{\frac{x-1}{2}} - 1\right]$ where x is an integer such that $x^2 - 2y^2 = 1$ for some $y \in \mathbb{Z}$.

Proof. Suppose n is an integer and h is the diagonal of R. Then by Pythagoras theorem, we obtain the equation $n^2 + (n+r)^2 = h^2$. By writing the equation, we get $k^2 - 2h^2 = -r^2$ where $k = 2n + r$. Lemma (2.2) shows that $k = r\sqrt{\frac{x-1}{2}}$ where $x^2 - 2y^2 = 1$ for some $y \in \mathbb{Z}$. Now we claim that $k = r\sqrt{\frac{x-1}{2}}$ implies n is an integer. It is enough to prove that $k - r$ is even. Now, $k - r = r\left(\sqrt{\frac{x-1}{2}} - 1\right)$. Therefore, $k - r$ is even if $\sqrt{\frac{x-1}{2}} - 1$ is even. Theorem (3.1) shows that $\sqrt{\frac{x-1}{2}}$ is odd and so $\sqrt{\frac{x-1}{2}} - 1$ is even. Hence $n \in \mathbb{Z}$ and n is of the form $n = \frac{r}{2}\left[\sqrt{\frac{x-1}{2}} - 1\right]$.

Suppose $n = \frac{p}{q} \in \mathbb{Q}\backslash\mathbb{Z}$. Then by Pythagoras theorem, we have $h^2 = \frac{2p^2+q^2r^2+2pqr}{q^2}$. Since h is an integer, its square is also an integer. This gives us $2p^2 + 2pqr = q^2\,(k_1 - r^2)$ for some $k_1 \in \mathbb{Z}$. This leads to the fact that q^2 divides $2p(p+qr)$. As p and q are relatively prime, it is clear that $2(p+qr) = k_2 q^2$ for some $k_2 \in \mathbb{Z}$. This gives $k_2 = \frac{2p}{q^2} = \frac{2r}{q^2}$ but this is not an integer, a contradiction.

Theorem 4.2. Let R be a rectangle with sides $n, n-r$ where $n \in \mathbb{Q}$ and $r \in \mathbb{N}$ such that $r > 1$. If the diagonal of R is an integer, then $n = \frac{r}{2}\left[\sqrt{\frac{x-1}{2}} + 1\right]$ where x is an integer such that $x^2 - 2y^2 = 1$ for some $y \in \mathbb{Z}$.

Proof. The proof is the same as in theorem (4.1).

Theorem 4.3. Let R be a rectangle with sides $n, n+r$ where $n \in \mathbb{Q}, r \in \mathbb{N}\backslash\{1\}$. If the diagonal of R is a rational number of the form $\frac{p}{q}, (p,q) = 1$, then $n = \frac{r(a^2-b^2)}{b^2-a^2+2ab}$ for some $a, b \in \mathbb{Z}$.

Proof. Let h be the diagonal of R. If $n \in \mathbb{Z}$, then $n^2 + (n+r)^2 = h^2 \in \mathbb{Z}$ and so $h \in \mathbb{Z}$. Thus no required h exists when $n \in \mathbb{Z}$. If $n = \frac{u}{v} \in \mathbb{Q}, n^2 + (n+r)^2 = h^2$ implies $u^2 + (u+vr)^2 = \left(\frac{pv}{q}\right)^2$. Since $u, v \in \mathbb{Z}$, we must have $q|v$. Take $v = kq$ for some $k \in \mathbb{Z}$. Replacing this in the last expression, we get an equation of the form $x^2 + y^2 = z^2$ where $x = u, y = u + kqr, z = pk$. This equation has to be solved over $\mathbb{Z}$. The solution to this equation is

$$u = \alpha\,(a^2 - b^2)$$

$$u + kqr = 2\,\alpha\, ab$$

$$pk = \alpha(a^2 + b^2)$$

where $\alpha, a, b \in \mathbb{Z}$. This gives $v = \frac{\alpha}{r}(b^2 - a^2 + 2ab)$ and so $n = \frac{r(a^2-b^2)}{b^2-a^2+2ab}$.

Theorem 4.4. Let R be a rectangle with sides $n, n-r$ where $n \in \mathbb{Q}, r \in \mathbb{N}\backslash\{1\}$. If the diagonal of R is a rational number of the form $\frac{p}{q}, (p,q) = 1$, then $n = \frac{r(a^2-b^2)}{a^2-b^2-2ab}$ for some $a, b \in \mathbb{Z}$.

Proof. The proof is the same as in the theorem (4.3) with $u = \alpha(a^2 - b^2)$ and $u - kqr = 2\alpha ab$.

5. Almost and Pseudo Almost Equilateral rational Rectangles with Area Equals Perimeter

In this section, we collect almost and pseudo almost equilateral rational rectangles with an area equal perimeter. For that purpose, we employ the elementary quadratic Diophantine equation.

Theorem 5.1. One cannot find a rectangle R with sides $n, n+1$ where $n \in \mathbb{Q}$ such that area equals the perimeter.

Proof. It is clear that the area of R is $n^2 + n$ and the perimeter of R is $4n + 2$. Equating this we obtain the equation $n^2 - 3n - 2 = 0$. Solving this quadratic equation we obtain two irrational roots. That is, we are not able to find a rational n.

Theorem 5.2. One cannot find a rectangle R with sides $n, n-1$ where $n \in \mathbb{Q}$ such that area equals the perimeter.

Proof. As in the theorem (5.1), we get the equation $n^2 - 5n + 2 = 0$. This equation also provides us with two irrational roots.

Theorem 5.3. The number of rectangles R with sides $n, n+r$ where $n \in \mathbb{Q}, r \in \mathbb{N}\backslash\{1\}$ such that area equals the perimeter is exactly one.

Proof. Here the area of R is $n^2 + nr$ and perimeter is $4n + 2r$. Equating this we get the equation $n^2 + (r-4)n - 2r = 0$. The roots of this equation are $n = \frac{4-r\pm\sqrt{r^2+16}}{2}$, which is rational if $r^2 + 16 = s^2$ for some $s \in \mathbb{Z}$. Rewriting this we get $(s+r)(s-r) = 16$. By writing 16 as a product of two numbers and comparing it with the other side, we see that the only suitable value of r is $r = 3$. This leads to the values $n = 3, -2$. As n cannot be negative, we conclude that $n = 3$.

Theorem 5.4. The number of rectangles R with sides $n, n-r$ where $n \in \mathbb{Q}, r \in \mathbb{N}\backslash\{1\}$ such that area equals the perimeter is exactly one.

Proof. As in the theorem (5.3), here we get $n = \frac{4+r\pm\sqrt{r^2+16}}{2}$ and for a rational n, we must have only $r = 3$. This gives $n = 6$.

Remark 2. From theorems (5.3) and (5.4), one can notice that there is exactly one pseudo almost equilateral rational rectangle exists in a way that its area coincides with the perimeter. Its sides are 6,3.

6. Conclusion

In this paper, all almost and pseudo almost equilateral rational rectangles having rational diagonal and area same as perimeter are collected with the help of Diophantine equations and their solutions. Taking this as an initial idea, one may think of another relation on rectangles and study their properties.

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