A COMPARISON WITH GREEN'S RELATION AND OUR MORE SPECIAL RELATIONS

ABSTRACT

David Mclean[2] has managed to decomposition of a band into additional specialty ensembles. He created a band by semilattice union of rectangular bands. We have made an effort to create a Near idempotent semigroup as a union of more specific Near idempotent semigroups in response to this conclusion. For this reason, we changed the Mclean relations L and R into λ and ρ before moving on to define δ as C, which is also the same as $\rho \circ \lambda$, as inspired by Green's $\mathcal{D}$[3].

Keywords: *decomposition of a band, semilattice, rectangular band, left/right singular near idempotent semigroup.*

Introduction:

Our introduction of relations λ, ρ on a near idempotent semigroup is inspired by McLean's relations L and R defined on a band respectively. This exercise's goal is to break down a generic Near idempotent semigroup decomposition, more specialised one. Each δ-class is a near idempotent semigroup into rectangular near idempotent semigroup, and each λ(ρ) -class is a left/right singular near idempotent semigroup. Finally, we demonstrate that the relations λ, ρ, δ and ξ are nothing more than the extensions of the relations of green $\mathcal{L}, \mathcal{R}, \mathcal{D}$ and $\mathcal{H}$..

Preliminaries and Notations

Definition -Idempotent

If ee = e (e^2 = e), then element e of a semigroup S is referred to be an idempotent element of S.

Zero elements and one-sided identities are idempotent.

The opposite is typically untrue.

Definition

let S be a semigroup and 'a' an element of S. 'a' is said to be a **near – idempotent element** of S if $xa^2y = xay$ for all x, y in S.

A semigroup S is called a **near idempotent semigroup** if every element of S is near idempotent element of S.

In any semigroup S, the left (right, two – sided) identity elements and the left (right, two – sided) zero elements are idempotents.

Definition - Band

If every element of a semigroup S is idempotent we shall say that S itself is idempotent , or that S is a **band**. (Klein – Barmen 1940)

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Definition - Semi – lattice

A commutative band is called a **semi – lattice.**

Definition - rectangular band

Let a and b be any two non – empty sets. Then, the system S = (A × B, *) where (a,b) * (a', b') = (a, b')

For all a, a' in A and b, b' in B is a band. It is called a **rectangular band** or an **anti-commutative band**.

Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{D}$ and $\mathcal{H}$.

Let S^1 for each semigroup S be a semigroup generated from S by adjoining an identity if S does not already contain an identity, and let S^1 Equal S otherwise. The equivalence relations on the set S that Green first developed are known as the "Green's relations of S."

According to such definitions, the L - relation as follows.

For any $a, b \in S$, $a\ L\ b$ if and only if $S^1a = S^1b$, or equivalently, $a\ L\ b$ if and only if $a = xb$ and $b = ya$ for some $x, y \in S^1$.

Dually, the R - relation defined as follows.

$a\ R\ b$ if and only if $aS^1 = bS^1$, or equivalently, $a\ R\ b$ if and only if $a = bx$ and $b= ay$ for some $x, y \in S^1$.

Moreover, the J - relation defined as follows.

$a\ J\ b$ if and only if $S^1 aS^1 = S^1 bS^1$,

or equivalently, $a\ J\ b$ if and only if $a = xby$ and $b = uav$ for some $x, y, u, v \in S^1$.

Finally, we define $H = L \cap R$ and $D = L \circ R$, where $\circ$ is the composition of relations. Since the relations L and R commute, it follows that $L \circ R = R \circ L$.

Special relations similar to Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{D}$ and $\mathcal{H}$.

We first define the dual relations λ and ρ on a near idempotent semigroup in the following.

Definition

Let S be a near- idempotent semigroup and a and b, elements of S. We define the relations λ and ρ on S as follows:

a λ b if and only if xaby = xay and xbay = xby for all x, y ∈ S

a ρ b if and only if xaby = xby and xbay = xay for all x, y ∈ S.

Both λ and ρ turn out to be equivalence relations on S. It is easy to check that λ is a right congruence and ρ is a left congruence relation on S.

Lemma

Let S be a near idempotent semigroup. Then the relation λ is an equivalence relation on S.

Proof $xa^2y = xay$ for all x, y , a ∈ S, by the definition of near- idempotent semigroup, so that a λ a for all a in S.

Hence λ is reflexive.

Let a λ b.

Then xaby = xay and xbay = xby for all x, y ∈ S which also implies b λ a.

Hence λ is symmetric.

Let a λ b and b λ c

Then for all x, y ∈ S,

we have, xaby = xay, xbay = xby and xbcy = xby, xcby = xcy

Hence xacy = x a cy = x ab cy = x a bc y = x ab y = xay for all x, y ∈ S

Similarly,

xcay = x cb a y = x c ba y = x c b y = xcy for all x, y ∈ S,

Which implies a λ c. Hence λ is transitive.

Thus λ is an equivalence relation on S.

Dually, We can prove that ρ is an equivalence relation on the near - idempotent semigroup S.

We now prove that λ is a right congruence and ρ is a left congruence relation on S.

The following theorem shows that λ is a right congruence relation on S.

Lemma

Let S be a near – idempotent semigroup. Let a λ b. Then ac λ bc for all c ∈ S

Proof

Let a λ b where a, b ∈ S.

We claim that for any c ∈ S, ac λ bc.

a λ b ⇒ xaby = xay and xbay = xby for all x, y ∈ S

Then for all x, y ∈ S, we have

xac bcy = xa cbc y = xab cbc y = x a $(bc)^2$ y = xabcy (by the definition of S) = xab c y = xacy and

x bc ac y = x b cac y = x ba cac y = xb $(ac)^2$ y = x b ac y (by the definition of S) = x ba c y = x bc y

Leading to ac λ bc for all c ∈ S, Hence λ is a right congruence on S.

Dually, ρ is a left congruence relation on S.

We now consider the composition of the two relations λ and ρ and prove that . λ ∘ ρ = ρ ∘ λ.

Lemma:

If S is a near- idempotent semi group, then λ ∘ ρ = ρ ∘ λ in S.

Proof :

We first prove that λ ∘ ρ ⊂ ρ ∘ λ .

Let a λ ∘ ρ b.

Then there exists c ∈ S such that a λ c and c ρ b.

a λ c ⇒ xacy = xay and xcay = xcy for all x, y ∈ S.c ρ b ⇒ xcby = xby and xbcy = xcy for all x, y ∈ S.

Choose d = acb.

Then for all x, y in S,

xady = x a acb y = x a^2 cb y = x a cb y = x acb y = xdy

xday = x acb ay = x ac bay

= x a b ay

= xab a y

= xab ac y

= xacy (since ρ is a left congruence ab ρ ac)

But xacy = xay,

so that finally we get xday = xay

Therefore a ρ d

Similarly,

xdby = xacbby = $xacb^2y$ = xacby = xdy for all x, y in S.

xbdy = x b acb y = xba cb y = xba b y = x b aby = x cb aby = xcby

(since λ is a right congruence ab λ cb)

But xcby = xby,

so that we get xbdy = xby

Hence d λ b

Thus a ρ d, d λ b so that a ρ ∘ λ b

This gives λ ∘ ρ ⊂ ρ ∘ λ

By a similar argument,

We can prove that

ρ ∘ λ ⊂ λ ∘ ρ Thus we get λ ∘ ρ = ρ ∘ λ

We now define the relation δ on S as follows

Definition:

Let S be a near- idempotent semigroup. Let a, b ∈ S. we define δ = λ ∘ ρ. In other words, a δ b if and only if there exists c ∈ S Such that a λ c and c ρ b.

We have already proved that λ ∘ ρ = ρ ∘ λ. Hence w e can write a λ ∘ ρ b or a ρ ∘ λ b instead of a δ b.

We now prove that δ is an equivalence relation on the near – idempotent semigroup S.

Lemma

Let S be a near idempotent semigroup. δ is an equivalence relation on S.

Proof:

For all a in S, a λ a and a ρ a since λ and ρ are reflexive,

so that a λ ∘ ρ a, which means a δ a.

Hence δ is reflexive.

a δ b ⇒ a λ ∘ ρ b

There exists u ∈ S such that a λ u and u ρ b

There exists u ∈ S such that b ρ u and u λ a since λ and ρ are symmetric

⇒ b ρ ∘ λ a ⇒ b δ a

(since λ ∘ ρ = ρ ∘ λ = δ)

Hence δ is symmetric.

a δ b, b δ c

⇒ There exist u, v ∈ S such that a λ u and u ρ b; b λ v and v ρ c

Since u ρ b and b λ v

we have u $\rho \circ \lambda$ v

We have u $\lambda \circ \rho$ v since $\lambda \circ \rho = \rho \circ \lambda$

Thus there exists $w \in S$ such that u λ w and w ρ v

a λ u and u λ w so that a λ w;

w ρ v and v ρ c so that w ρ c

Therefore a $\lambda \circ \rho$ c

i.e, a δ c. Thus δ is transitive.

Hence δ is an equivalence relation on the near- idempotent semigroup S.

Conclusion

We demonstrate that the relations π and δ coincide on the near-idempotent semigroup S by defining the relation π on S in a manner similar to David McLean's relation P defined on a band[22]. In contrast to Green's, our relationship is more unique.

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