INTEGRAL SOLUTIONS OF QUADRATIC DIOPHANTINE EQUATION WITH TWO UNKNOWNS $11(\theta^2+\Omega^2)=2(12\theta\Omega-1)$

ABSTRACT

The Quadratic Diophantine equation $11(\theta^2+\Omega^2)=2(12\theta\Omega-1)$ *interpreting a conic is reviewed for its relevant integer lattice points. The recurrence relations amidst the solutions are also deduced.*

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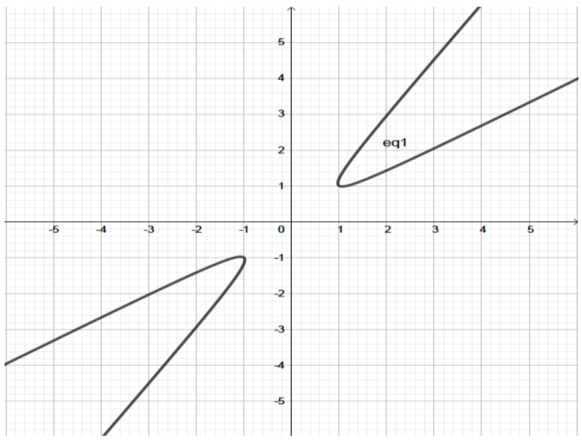
Introduction :

Integers are the main focus of Number Theory, an ancient discipline of mathematics. In honour of Diophantus of Alexandria, who made significant contributions to Number Theory by resolving algebraic equations, a significant component of Number Theory is named "Diophantine equations". Equations with more than one variable and roots that must be integers are known as diophantine equations. A few examples of Diophantine equation are Pythagorean, Pellian and Fermat's equations [1,2,3]. A crucial component of research is the solution of Diophantine equations, although there is no universal method for doing so. The quadratic Diophantine equation $11(\theta^2+\Omega^2)=2(12\theta\Omega-1)$ is taken into consideration in this work, and a few theorems are used to discover the answers. Additionally, the relations between the solutions' recurrence are also adjoined.

Figure 1: Graphical Representation of the Equation



Theorem 1 :

Let $\tilde{\tau}$ be the Diophantine equation in (3) . Then

(a) The square root of the coefficient 23 taken from (3) can be expanded as a continued fraction $[4;\overline{1,3,1,8}]$

(b) The primary solution of $\sigma^{*2}-23\zeta^{*2}=1$ is $(\sigma_1^*,\zeta_1^*)=(24,5)$.

(c) Define the string $\{(\sigma_n^*,\zeta_n^*)\}$, where $\begin{pmatrix}\sigma_n^*\\ \zeta_n^*\end{pmatrix}=\begin{pmatrix}24 & 115\\ 5 & 24\end{pmatrix}^n\begin{pmatrix}1\\ 0\end{pmatrix}$ for $n\geq 1$.

Then (σ_n^*,ζ_n^*) is a solution of $\tilde{\tau}$.

(d) The points (σ_n^*,ζ_n^*) caters to the relations $\sigma_n^*=24\sigma_{n-1}^*+115\zeta_{n-1}^*$ and $\zeta_n^*=5\sigma_{n-1}^*+24\zeta_{n-1}^*$ for $n\geq 2$.

(e) The points (σ_n^*,ζ_n^*) satisfy the recurrence relations
$\sigma_n^*=48\sigma_{n-1}^*-\sigma_{n-2}^*$;
$\zeta_n^*=48\zeta_{n-1}^*-\zeta_{n-2}^*$ for $n\geq 4$.

Proof:

(a) The continued fraction expansion of $\sqrt{23}=4+\left(\sqrt{23}-4\right)$

$$=4+\frac{1}{\frac{1}{\sqrt{23}-4}}$$

$$=4+\frac{1}{\frac{\sqrt{23}+4}{7}}$$

$$=4+\frac{1}{1+\frac{1}{\frac{1}{\frac{\sqrt{23}-3}{7}}}}$$

$$=4+\frac{1}{1+\frac{1}{\frac{7}{\sqrt{23}-3}}}$$

$$=4+\frac{1}{1+\frac{1}{3+\frac{1}{\frac{2}{\sqrt{23}-3}}}}$$

$$=4+\frac{1}{1+\frac{1}{3+\frac{1}{\frac{\sqrt{23}+3}{7}}}}$$

$$=4+\frac{1}{1+\frac{1}{3+\frac{1}{1+\frac{1}{\frac{7}{\sqrt{23}-4}}}}}$$

$$=4+\frac{1}{1+\frac{1}{3+\frac{1}{1+\frac{1}{\sqrt{23}+4}}}}$$

$$= 4+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{8+\sqrt{23}-4}}}}$$

∴ The continued fraction expansion of $\sqrt{23}$ is $[4;\overline{1,3,1,8}]$.

(b) Since $\sigma_1^{*2}-23\zeta_1^{*2} = (24)^2-23(5)^2$

$= 1$

is satisfied for the values $(\sigma_1^*,\zeta_1^*)=(24,5)$, (b) is proved.

(c) The approach of Mathematical induction is employed here. Consider

$\binom{\sigma_n^*}{\zeta_n^*}=\begin{pmatrix}24 & 115\\ 5 & 24\end{pmatrix}^n\binom{1}{0}$……….(4).

Let n=1,we get $(\sigma_1^*,\zeta_1^*)=(24,5)$ which is a solution of $\tilde{\tau}$. Assume that the result is ture for n-1. ie, for the Diophantine equation

$\tilde{\tau}: \sigma_{n-1}^{*2}-23\zeta_{n-1}^{*2}=1$ …….(5).

ie, $\binom{\sigma_{n-1}^*}{\zeta_{n-1}^*}=\begin{pmatrix}24 & 115\\ 5 & 24\end{pmatrix}^{n-1}\binom{1}{0}$.

The general solution(σ_n^*,ζ_n^*) can be expressed as

$\binom{\sigma_n^*}{\zeta_n^*}=\begin{pmatrix}24 & 115\\ 5 & 24\end{pmatrix}\binom{\sigma_{n-1}^*}{\zeta_{n-1}^*}$……(6).

To check for its solvability, we have

$\sigma_n^{*2}-23\zeta_n^{*2} = (24\sigma_{n-1}^*+115\zeta_{n-1}^*)^2-23(5\sigma_{n-1}^*+24\zeta_{n-1}^*)^2$

$= \sigma_{n-1}^{*2}-23\zeta_{n-1}^{*2}$

$= 1$

Thus the result is valid for n so that (σ_n^*,ζ_n^*) is a solution of $\tilde{\tau}$. Hence (c) is proved.

(d) From (6), we find that

$\sigma_n^*=24\sigma_{n-1}^*+115\zeta_{n-1}^*$;

$\zeta_n^*=5\sigma_{n-1}^*+24\zeta_{n-1}^*$ for $n\geq 2$ ……..(7)

(e) We have to prove that σ_n^* and ζ_n^* satisfy the given recurrence relations.

From (7), assuming $(\sigma_0^*,\zeta_0^*)=(1,0)$,the consecutive solutions are found to be

$\sigma_1^*=24, \sigma_2^*=1151, \sigma_3^*=55224$, $\sigma_4^*=2649601$ and $\zeta_1^*=5, \zeta_2^*=240$, $\zeta_3^*=11515$, $\zeta_4^*=552480$ and so on. It has been found that all these values satisfy the recurrence relations

$\sigma_n^*=48\sigma_{n-1}^*-\sigma_{n-2}^*$; $\zeta_n^*=48\zeta_{n-1}^*-\zeta_{n-2}^*$.

Theorem 2:

Identify a progression $\{(\theta_n,\Omega_n)\}$ of positive integers by $(\theta_1,\Omega_1)=(29,19)$ and $\theta_n=29\sigma_{n-1}^*+139\zeta_{n-1}^*$; $\Omega_n=19\sigma_{n-1}^*+91\zeta_{n-1}^*$, where $\{(\sigma_n^*,\zeta_n^*)\}$ is a sequence of positive solutions of $\sigma^{*2}-23\zeta^{*2}=1$. Then the solution (θ_n,Ω_n) satisfies the following results.

(a) $\theta_{n+1}=24\theta_n+115\Omega_n$;

$\Omega_{n+1}=5\theta_n+24\Omega_n$.

(b) $\theta_n=48\theta_{n-1}-\theta_{n-2}$;

$\Omega_n=48\Omega_{n-1}-\Omega_{n-2}$.

Proof :

(a) We have perceived that

$\theta_{n+1}+\Omega_{n+1}\sqrt{d}=(\sigma_1^*+\zeta_1^*\sqrt{d})(\theta_n+\Omega_n\sqrt{d})$ is a solution of the general Pell equation $\Omega^2=d\theta^2+1$. Hence

$\theta_{n+1}=\sigma_1^*\theta_n+\zeta_1^*\Omega_n d$ and $\Omega_{n+1}=\zeta_1^*\theta_n+\sigma_1^*\Omega_n$.

Thus $\theta_{n+1}=24\theta_n+115\Omega_n$;

$\Omega_{n+1} = 5\theta_n + 24\Omega_n$,
since $\sigma_1^* = 24$ and $\zeta_1^* = 5$.

(b) Applying the equalities,
$\theta_n = 29\sigma_{n-1}^* + 139\zeta_{n-1}^*$; $\Omega_n = 19\sigma_{n-1}^* + 91\zeta_{n-1}^*$, we come across, by generation on n, that $\theta_n = 48\theta_{n-1} - \theta_{n-2}$;
$\Omega_n = 48\Omega_{n-1} - \Omega_{n-2}$.

Conclusion

Quadratic Diophantine equations with two unknown have been studied and solved using a variety of techniques. One might look for several types of solution patterns.

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