PERFECT MATCHING OF AN UNDIRECTED SPARSE GRAPH BASED ON GEOMETRIC MULTIPLICITY

ABSTRACT

One of the important concepts of Graph Theory is Matching Theory. Several concepts on Matching Theory have been dealt in [1, 2 & 3]. The technique of maximum matching on directed graphs has been studied in [1]. In this paper, a new approach, for finding maximum matching of an undirected graph based on largest geometric multiplicity of Eigen values using exact controllability network, is studied. The adjacency matrix of an undirected sparse graph has a relation with the exact controllability network for finding the maximum matched nodes and the corresponding matched edges using largest geometric multiplicity of Eigen values. The new method is illustrated using an undirected sparse graph for finding perfectly matched edges for N nodes based on Geometric multiplicity. Further, the concept is used to obtain a perfectly matched line graph with size N and the line graph if connected parallel forms a perfectly matched undirected square grid sparse graph of size 4. In general, the square grid graph of size 2n+2 is perfectly matched for all $n \geq 1$ *. It will be interest to further study on their properties.*

Keywords: *Graph Theory, Matching, Maximum Matching, Geometric Multiplicity, sparse graph, Line graph, Grid sparse graph.*

1. Introduction:

One of the important concepts of Graph Theory is Matching Theory. It is not only used to study the structure of a graph also has a close relation with practical problems in Network work flow, Interns to Hospital Residency programmes, Resource allocation including balancing the traffic load etc. To solve practical problems maximum matching should be used.

Under this topic we discuss the method to find maximum matching of an undirected sparse graph using Largest Geometric Multiplicity of Eigen values. In an undirected graph the maximum set of edges without common nodes is known as maximum matching. Maximum Matching nodes can be obtained using the largest geometric multiplicity of its adjacency matrix. The matching edges corresponding to the matching nodes are obtained through fundamental transformation. The maximum matching of an undirected sparse graph is discussed under this topic with theorems. The basic idea of this method is obtained from the exact controllability for Sparse Networks [3]. The Definitions and working rules are discussed before proving the theorems.

HEMALATHA S

Department of Mathematics, Shrimathi Devkunvar Nanalal Bhatt Vaishnav College for Women, Chrompet, Chennai – 600 044, Tamil Nadu.

JINI J*

Department of Mathematics, Kings Engineering College, Sriperumbudur, Chennai – 602 117, Tamil Nadu.

** Research Scholar, Shrimathi Devkunvar Nanalal Bhatt Vaishnav College for Women, Affiliated to University of Madras, Chrompet, Chennai-600 044, Tamil Nadu.*

BALA S

Department of Mathematics, S.I.V.E.T College, Gowrivakkam, Chennai – 600 073, Tamil Nadu.

2. Notation and Preliminaries

2.1. Notation: $G_u = (V, E)$, where $G_u$is the Undirected graph, V be the Nodes (non empty finite set of elements) and E is the edges (finite set of ordered pairs of different nodes).

Example: An Undirected graph $G_u$with 4 nodes and 6 edges.

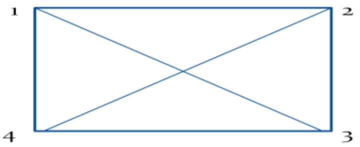

Fig 1. Undirected graph

Set of vertices,$V = \{ 1,2,3,4\}$ and Set of edges E= {(1,2), (2,3), (3,4), (1,4), (1,3), (2,4)}

Note: The size of undirected graph is the number of nodes in the graph G_u and it is denoted by $\left| G_u \right| = N$.

The definition of Matching, Maximum matching of an undirected graph, the concept of Geometric multiplicity of Eigen values are generalized in the following Preliminaries

2.2. Definition (Geometric Multiplicity):

In general, from [1] for an undirected graph the largest number of linearly independent Eigen vectors corresponding to an Eigen value is known as its Geometric Multiplicity.

2.3. Definition (Matching of an undirected graph): The independent set of edges where no two share a node is Matching M of an undirected graph. If the node is incident to an edge, then it is matched, otherwise it is unmatched.

2.4. Definition (Maximum Matching of an undirected graph): The Matching of Maximum cardinality among all matching is known as Maximum Matching $M^*$and if all the nodes are matched then the Maximum Matching M^* is a perfect matching.The edges corresponding to the matched nodes are the matched edges.

2.5. Definition (Sparse graph): A graph is called sparse if the number of edges is much less than the possible number of edges.

Note: An Undirected graph can have at most $\frac{n(n-1)}{2}$ edges.

2.6. Largest Geometric Multiplicity:

As proved in [1] the minimum number of driver's node N_D of an undirected graph is obtained by the largest geometric multiplicity $\mu(\lambda_j)$ of the Eigen value $\lambda_j$of A_j

$$\mu(\lambda_j) = \dim V_{\lambda_j}$$
$$= N - rank\,\{\lambda_j I_N - A\}$$

Where $\lambda_j (j = 1, 2, 3, \ldots, N)$ represents the distinct Eigen values of A and I_N is the unit matrix with the same as A.

$$N_D = \max\{\mu\,(\lambda_j)\}.$$

3. Maximum matching of an Undirected Graph based on Largest Geometric Multiplicity

From the condition explained in [1] for any undirected graph in general $\mu(\lambda_j)$ is the largest geometric multiplicity of the Eigen values λ_j . Let matrix A' be the column canonical form of matrix $\lambda_j I_N - A$. Then Linearly independent rows in A' are matched nodes and linearly dependent rows are unmatched nodes.

Theorem3.1. For all $n \geq 0$ prove that Sparse graph of size N is perfectly matched with $\frac{N}{2} = (4 + 2n)(1)^n$ matched edges and $9(2)^n$ number of different perfectly matched graphs using Geometric multiplicity of its Eigen values.

PERFECT MATCHING OF AN UNDIRECTED SPARSE GRAPH BASED ON GEOMETRIC MULTIPLICITY

Proof: Consider an undirected Sparse graph of any size N=4n (n>1) without self-loops. Then the expected Eigen values λ_j(j=1,2,3……N) of the Adjacency matrix A will be zero since all the rows of the matrix A are Linearly independent.

$$E(\lambda) = \frac{1}{N}\sum_{j=1}^{N}\lambda_j = \frac{1}{N}\sum_{j=1}^{N}a_{jj} = 0 \quad (1)$$

If the rows are linearly independent, then by (2.6) the Geometric Multiplicity $\mu(\lambda_j)$ corresponding to the Eigen values $\lambda_j = 0$ is also zero.

$$rank\,\{\lambda_j I_N - A\} = rank(-A)$$

$$= rank(A) = N$$

(N number of non-zero rows)

Therefore, $\mu(\lambda_j) = N - rank\,\{\lambda_j I_N - A\}$

$$= N - N = 0$$

Thus, all the nodes (N) are matched nodes, since according to maximum matching of an undirected graph the largest Geometric multiplicity $\mu(\lambda_j)$ is related to unmatched nodes and if it is zero all the nodes are matched.

The above proof is illustrated by an undirected sparse graph

Case (i) Consider an undirected sparse graph with 8 vertices and 12 edges

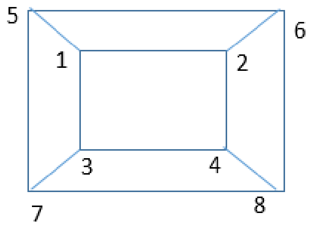

Fig 2. Undirected graph with 8 vertices and 12 Edges

Matrix Form: The Adjacency matrix is, $A \in R_{N\times N}$ of a graph (V, E) with vertices $V = \{v_1, v_2, \ldots, v_N\}$ is given by

$$a_{ij} = \begin{cases} 1, & if\ (v_i, v_j) \in E \\ 0, & otherwise \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

It is a sparse graph without self-loops and from equation (1), it is known that the expected mean Eigen value is zero (i.e.) $\lambda_j = 0$.

Largest Multiplicity of a Sparse Graph

The Eigen value zero corresponds to the largest geometric multiplicity. Each Eigen value of A has geometric multiplicity, that is exactly 0 and each row in A is linearly independent. The nodes corresponding to the linearly independent rows are matched nodes with numerical

$$rank(\lambda_m I_N - A) = rank\,(-A)$$
$$= rank(A) = 8$$

Therefore, all nodes are perfectly matched.

Perfectly Matched Edges

The Perfectly Matched Edges for the 8 matched nodes are 4, which are shown in red lines. It is given in a way that no two edges share a node. Fig 3. shows that there are 9 different perfectly matched graphs for an undirected sparse graph with 8 nodes and 12edges.

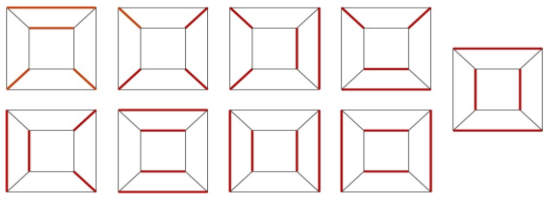

Fig 3. Perfectly matched edges (Sparse graph)

Case (ii) Consider an undirected sparse graph with 12 vertices and 20 edges

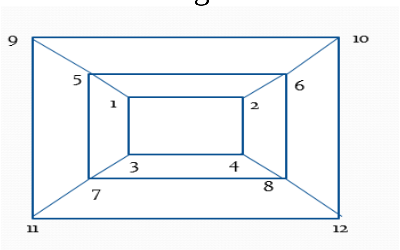

Fig 4. Undirected graph with 12 nodes and 20 edges

Matrix Form: The Adjacency matrix is, $A \in R_{N \times N}$ of a graph (V, E) with vertices$V = \{v_1, v_2, \ldots, v_N\}$ is given by

$$a_{ij} = \begin{cases} 1, & if\ (v_i, v_j) \in E \\ 0, & otherwise \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

It is a sparse graph without self-loops and from equation (1), it is known that the expected mean Eigen value is zero (i. e) $\lambda_j = 0$.

Largest Multiplicity of a Sparse Graph

The Eigen value zero corresponds to the largest geometric multiplicity. Each Eigen value of A has geometric multiplicity, that is exactly 0 and each row in A is linearly independent. Then the nodes corresponding to the linearly independent rows are matched nodes with numerical

$$rank(\lambda_m I_N - A) = rank\ (-A)$$
$$= rank(A) = 12$$

Therefore, all nodes are perfectly matched.

Perfectly Matched Edges

The Perfectly Matched Edges for the 12 matched nodes are 6, which is shown in red lines. It is given in a way that no two edges share a node. Fig 5. shows that there are 9+9=18 different perfectly matched graphs for an undirected sparse graph with 12 nodes and 20 edges.

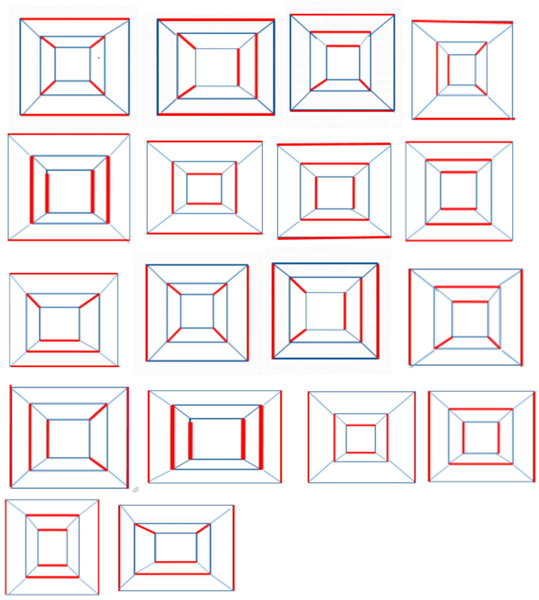

Fig 5.Perfectly matched edges (Sparse graph)

Case (iii) Consider an undirected sparse graph with 16 vertices and 28 edges

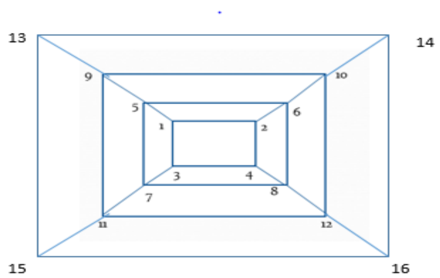

Fig 6. Undirected graph with 16 nodes and 28 edges

From the Adjacency matrix of Case (i) and Case (ii) it is known that undirected sparse graph in Fig.6 has 16 linearly independent rows. Then by Theorem.3.1 the expected mean Eigen value is zero

(i. e) $\lambda_j = 0$.

Largest Multiplicity of a Sparse Graph

The Eigen value zero corresponds to the largest geometric multiplicity. Each Eigen value of A has geometric multiplicity, that is exactly 0 and each row in A is linearly independent. The nodes corresponding to the linearly independent rows are matched nodes with numerical

$$rank(\lambda_m I_N - A) = rank\ (-A) = rank(A) = 16$$

Therefore, all nodes are perfectly matched.

Perfectly Matched Edges

The Perfectly Matched Edges for the 16 matched nodes are 8, which is shown in a way that no two edges share a node and there will be18+18=36 different perfectly matched graphs for the undirected sparse graph with 16 nodes and 28 edges.

Case (iv) Consider an undirected sparse graph with N=4n vertices and E=4(2n-1) edges

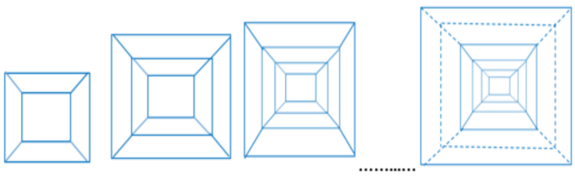

Fig 7. Undirected graph with N=4n nodes and E=4(2n-1) edges

For $n \geq 0$, considering an undirected sparse graph with N=8, 12,16, …….4(n+2) vertices and E=12, 20,28,…,4(2n+3) edges form an Adjacency matrix in which all the rows will be linearly independent. Then the expected Eigen value $\lambda_j = 0$ (j=1,2,3……N) and the largest Geometric multiplicity $\mu(\lambda_j)$ is zero if all the rows are linearly independent and it clearly shows that all the nodes (N) are matched since the largest geometric multiplicity is related to unmatched nodes.

Therefore, the maximum matching of an undirected sparse graph of size N (nodes) and E edges without self-loops based on its Largest Geometric multiplicity is perfectly matched.

The maximum number of matched edges for each size with initial value $a_0 = 4$ is given by,

For n =1, $a_1 = a_0 + 2 = 4 + 2 = 6$

For n =2, $a_2 = a_1 + 2 = 6 + 2 = 8$ and so on, which forms a Homogeneous Recurrence Relation,

$$a_n - a_{n-1} = 2,\ \ a_0 = 4 \text{ for all } n \geq 0 \qquad (1)$$

To find Homogeneous solution,

put $a_n = r^n$

$$r^n - r^{n-1} = 0$$

$$r^{n-1}(r-1) = 0$$

$$r^{n-1} \neq 0,\quad r = 1$$

The Homogeneous solution is given by,

$$a_n{}^{(h)} = k_1(1)^n$$

To find Particular solution, consider $f(n) = 2(1)^n$ then

$$a_n{}^{(p)} = A_0 n(1)^n \qquad (2)$$

Substituting (2) in (1) and equating the like coefficients on both sides gives,

$$A_0 n(1)^n - (A_0(n-1)(1)^{n-1}) = 2(1)^n$$

$$A_0 = 2$$

$$a_n{}^{(p)} = 2n(1)^n \qquad (3)$$

The General solution is,

$$a_n = a_n{}^{(h)} + a_n{}^{(p)}$$
$$= k_1(1)^n + 2n(1)^n \qquad (4)$$

Given $a_0 = 4$

(i.e.) for n=0, $a_0 = k_1(1)^0 + 2(0)(1)^0$

$$4 = k_1 + 0$$

$$k_1 = 4$$

Therefore, in general the maximum number of matched edges is,

$\frac{N}{2} = a_n = (4+2n)(1)^n$ for all $n \geq 0$.

Then the number of different perfectly matched graphs are 9, 18,36, . . . for each size.

Consider the sequence 9, 18, 36,72, … as $a_o, a_1, a_2, a_3, \ldots$ then by Generating function

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$$
$$= 9x^0 + 18x^1 + 36x^2 + 72x^3 + \cdots$$
$$= 9(1 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots + (2x)^n + \cdots)$$

$$\sum_{n=0}^{\infty} a_n x^n = 9\sum_{n=0}^{\infty} (2x)^n$$

Equating coefficients of x^n on both sides gives the solution of the Generating function, $a_n = 9(2)^n$, for all $n \geq 0$

Therefore, in general the number of different perfectly matched graphs is

$a_n = 9(2)^n$ for all $n \geq 0$

Theorem.3.2. Maximum matching of an undirected line sparse graph connected in parallel forms a perfectly matched square grid graph

Proof: By Theorem.3.1 if all the rows of an Adjacency matrix are linearly independent then the Geometric multiplicity $\mu(\lambda_j)$ is zero and all the nodes are matched nodes.

For n >1 consider an undirected line sparse graph with N=2n vertices and E=N-1 edges.

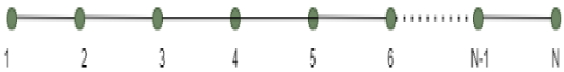

Fig 8. Undirected Line graph with N nodes and E edges

The Adjacency matrix is given by,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 1 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & 1 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

All the nodes of the line graph are matched since the rows of the Adjacency matrix is linearly independent.

Therefore, all the nodes are perfectly matched.

Perfectly Matched Edges

There are $\frac{N}{2} = 2, 3, \ldots, n$ perfectly matched edges for $N = 4, 6, \ldots, 2n$ nodes and $E = 3, 5, \ldots, 2n - 1$ edges

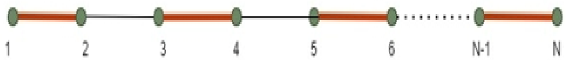

Fig 9. Perfectly matched edges (Line Sparse graph)

Now, connect an undirected Line sparse graph in parallel

Case (i) Consider an undirected two line sparse graph with N=4+4 vertices, E=3+3 edges and connected in parallel with 4 more edges.

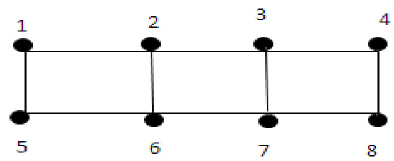

Fig 10.An undirected Line sparse graph connected in parallel with 8 vertices 10 edges

The Adjacency matrix is given by,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

All the nodes of the connected line graph are matched since the rows of the Adjacency matrix are linearly independent.

Therefore, all the nodes are perfectly matched.

Perfectly Matched Edges

There are 4 perfectly matched edges for 8 nodes and 5 different perfectly matched undirected connected line sparse graphs.

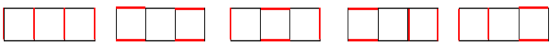

Fig 11. Perfectly matched edges

Case (ii) Consider an undirected three line sparse graph with N=4+4+4 vertices, E=3+3+3 edges and connected in parallel with 8 more edges.

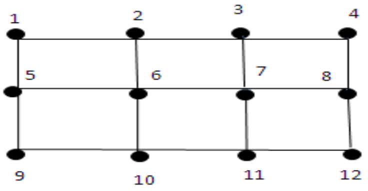

Fig.12. An undirected Line sparse graph connected in parallel with 12 vertices 17 edges

From Case (i) it is known that an undirected line connected sparse graph with adjacency matrix of 12 rows and 12 columns will be a linearly independent rows. Then based on Geometric multiplicity the Eigen value is zero and all the 12 nodes are matched nodes.

Therefore, All the nodes are perfectly matched

Perfectly Matched Edges

There are 6 perfectly matched edges for 12 nodes and 9 different perfectly matched undirected connected linear sparse graph

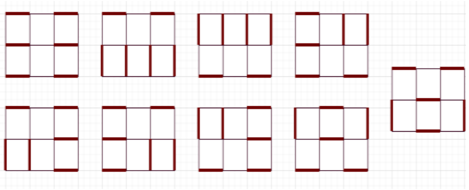

Fig 13. Perfectly matched edges

Case (iii) Consider an undirected four line sparse graph with N=4+4+4+4 vertices, E=3+3+3+3 edges connected in parallel with 12 more edges.

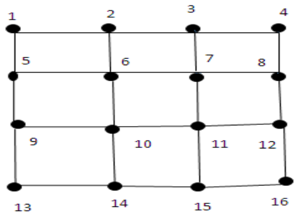

Fig.14. An undirected Line sparse graph connected in parallel with 16 vertices 24 edges

In fig.14.all the nodes of the undirected line sparse graph connected in parallel with 16 vertices and 24 edges are perfectly matched since in the Adjacency matrix all the rows will be linearly independent and it forms a square grid graph.

Perfectly Matched Edges

There will be 8 perfectly matched edges for 16 nodes and 24 different perfectly matched undirected connected line sparse graph.

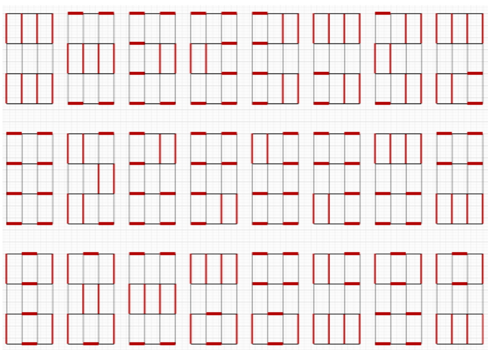

Fig 15. Perfectly matched edges

From case (iii) it is concluded that four line graph with 4 vertices connected in parallel has 16 vertices and 24 edges forms a square grid graph of size 4.

Theorem.3.3. For all $n \geq 1$ the square grid graph of size 2n+2 is perfectly matched and also area of the hut drawn on square grid graph is equal to the sum of matched edges and $n^2 - 1$.

Proof:

Case (i): From Theorem 3.2 it is proved that a square grid is formed with initial size 4(line graph) and vertices 16 by connecting the lines in parallel to each other. Now consider the square grid of size 4 with 16 vertices and 24 edges. The area of the square grid if each square of the grid is of side length 1 cm is 9cm^2.

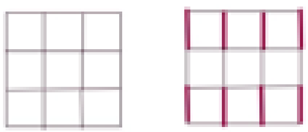

Fig 16. Square grid graph and perfectly matched graph of size 4

Fig.16. shows that a square grid graph of size 4 is a sparse graph and the adjacency matrix of the square grid is formed by 16 rows and 16 columns with rows linearly independent. Then by Geometric multiplicity all the nodes are matched since the rows are linearly independent and the square grid graph is perfectly matched with 8 matched edges

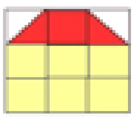

Fig 17. Hut drawn on square grid graph of size 4

Area of the Hut

= No. of full square +$\frac{1}{2}$ (No. of half square)

= (3x2) +1+$\frac{1}{2}(2) = 8\ cm^2$

Fig17. Shows that the area of the hut drawn on square grid graph of size 4 is equal to the sum of matched edges 8 of the perfectly matched grid graph and zero.

Case (ii) consider the square grid of size 6 with 36 vertices and 60 edges. The area of the square grid if each square of the grid is of side length 1 cm is 25cm^2.

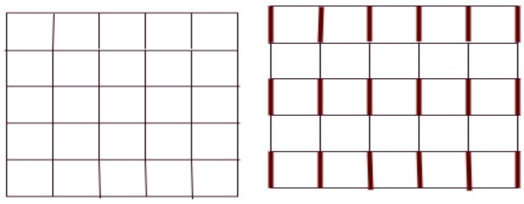

Fig 18. Square grid graph and perfectly matched graph of size 6

Fig.18. shows that a square grid graph of size 6 is a sparse graph and the adjacency matrix of the square grid is formed by 36 rows and 36 columns with rows linearly independent. Then by Geometric multiplicity all the nodes are matched since the rows are linearly independent and the square grid graph is perfectly matched with 18 matched edges

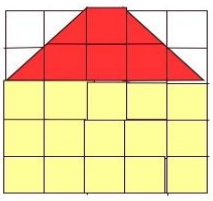

Fig 19. Hut drawn on square grid graph of size 6

Area of the Hut = No. of full square +$\frac{1}{2}$ (No. of half square)

= (5x3) +3+1+$\frac{1}{2}(4) = 21\ cm^2$

Fig19. Shows that the area of the hut drawn on square grid graph of size 6 is equal to the sum of matched edges 18 of the perfectly matched grid graph and 3.

Case (iii) Consider the size of square grid graph is formed as a sequence 4, 8, 16, 32, … (2n+2) by adding square of size 1 around the previous square graph.

By Mathematical induction it is prove that,

$$S(n) = 4 + 6 + 8 + \cdots + (2n + 2)$$

$$= n^2 + 3n \text{ for all } n \geq 1$$

Basic step: For n=1, S(1)=$(1)^2$+3(1)

4= 4

i.e., Basic step is true

Induction step: For n=k

Assume that it is true for n=K

i.e., $S(k) = 4 + 6 + 8 + \cdots + (2k + 2)$

$$= k^2 + 3k \text{ for all } k \geq 1$$

For n =k+1

$$S(k + 1) = 4 + 6 + 8 + \cdots + (2k + 2) + (2k + 4)$$

$$= k^2 + 3k + 2k + 4$$

$$= (k + 1)^2 + 3(k + 1)$$

i.e., Induction step is true

Therefore, S(n) is true for all $n \geq 1$

In general, for all $n \geq 1$ the size of the square grid graph is $S(n) = 2n + 2$ with vertices $S^2(n) = (2k + 2)^2$.

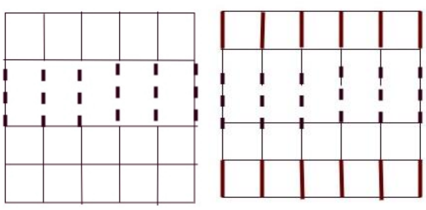

Fig 20. Square grid graph and perfectly matched graph of size 2n+2

Fig.20. shows that the square grid of size 2n+2 based on geometric multiplicity all the vertices are perfectly matched since the rows of the adjacency matrix are linearly independent with matched edges

$$\frac{S^2(n)}{2} = \frac{(2k+2)^2}{2} = 2(n + 1)^2$$

In real time application square grid graph is used to find area of any image drawn on the square grid.

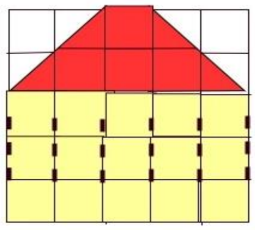

Fig 21. Hut drawn on square grid graph of size 2n+2

Area of the Hut = No. of full square + $\frac{1}{2}$(No. of half square)

= [(3,5,7,..., (2n+1)) x(2,3,4,..., (n+1)]+[1+3+5+...+(2n-1)]+$\frac{1}{2}(2n)$

$$= [(2n + 1) \times (n + 1)] + n^2 + \frac{1}{2}(2n)$$

$$=(2n^2 + n + 2n + 1) + n^2 + n$$

$$= (3n^2 + 4n + 1\ )cm^2$$

for all $n \geq 1$

For an example a Hut is drawn and shown that the area is equal to the sum of matched edges $2(n + 1)^2$ and $n^2 - 1$. The area of the square grid with each square side length 1cm is

$$(S(n)-1)^2 = (2n+1)^2 cm^2$$

4. Conclusion

In this topic the new method for finding maximum matched nodes of an undirected sparse graph based on largest geometric multiplicity of Eigen values which has a relation with exact controllability network is proved through adjacency matrix. The new method is proved by a theorem that, in general for an

undirected sparse graph based on Geometric multiplicity has a perfect matching if all the rows of its adjacency matrix are linearly independent. Also the same concept is used to prove that, in general if a perfectly matched undirected line sparse graph if connected in parallel form a perfectly matched undirected Grid graph. Finally, it is proved that in general the area of an image drawn on the square grid graph is equal to the sum of matched edges and $n^2 - 1$. In future, this concept can be expanded for some other undirected sparse graph.

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