CONNECTED SQUARE FREE DETOUR NUMBER OF A GRAPH

ABSTRACT

In this paper, we introduce and investigate the concepts of connected square free detour set of a graph G and the connected square free detour number $cdn_{\square f}(G)$ of G. Certain general properties of these concepts are studied. It is proved that all the end-vertices and cut-vertices of a connected graph G belong to every connected square free detour set of G. It is also proved that if T is a tree of order $n \geq 2$, then $cdn_{\square f}(T) = n$ and every branch of a connected graph G with cut-vertices contain an element of a connected square free detour basis. The connected square free detour basis of certain classes of graphs are determined. For any three positive integers α, β and n with $2 \leq \alpha < \beta \leq n$, there exists a connected graph G of order n with $dn_{\square f}(G) = \alpha$ and $cdn_{\square f}(G) = \beta$.

Keywords: *Connected detour number, connected square free detour set, connected square free detour number*

Introduction

In this article, a graph G is considered to be a non-trivial, finite, undirected and connected graph of order n with neither loops nor multiple edges. Let $D(x, y)$ be the longest path in G and an $x - y$ path of $D(x, y)$ is called $x - y$ detour. The parameters on detour concept was developed by Chartrand [1]. The connected detour concept was introduced by A. P. Santhakumaran and S. Athisayanathan [5]. The detour concept was extended to triangle free detour concept by S. Athisayanathan et al. [4].

For any two vertices u, v in a connected graph $G(V, E)$, the triangle free detour distance $D_{\Delta f}(u - v)$ is the length of a longest $u - v$ triangle free path in G. A $u - v$ path of length $D_{\Delta f}(u, v)$ is called a $u - v$ triangle free detour. A set $S \subseteq V$ *of* G is called a triangle free detour set of G if every vertex of G lies on a $u - v$ triangle free detour joining a pair of vertices of S. The triangle free detour number $dn_{\Delta f}(G)$ of G is the minimum order of its triangle free detour sets. This triangle free detour number was studied by S. Athisayanathan and S. Sethu Ramalingam [6]. This concept was extended to square free detour number by K. Christy Rani and G. Priscilla Pacifica [3]. A set $S \subseteq V$ of G is called a square free detour set if every vertex of G lies on a $u - v$ square free detour joining a pair of vertices of S.The square free detour number of G denoted by $dn_{\square f}(G)$ is defined as the minimum order of its square free detour sets.

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In this article, we introduce the connected square free detour number denoted by $cdn_{\square f}(G)$. We determine the connected square free detour number of some standard graphs.

The following theorems are used in the sequel. For the basic terminologies we refer to Chartrand [2].

Theorem 1.1 [3] Every end-vertex of a non-trivial connected graph G belongs to every detour set of G. Also, if the set S of all end-vertices of G is a square free detour set, then S is the unique square free detour basis for G.

Theorem 1.2 [3] Let $G = (V, E)$ be a complete graph K_n $(n \geq 2, n \neq 4)$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if S consists of any two adjacent vertices of G.

Theorem 1.3 [3] Let $G = (V, E)$ be an odd cycle C_n of order $n \geq 3$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if S consists of any two adjacent vertices of G.

Theorem 1.4 [3] Let $G = (V, E)$ be an even cycle C_n of order $n \geq 4$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if S consists of

(i) any two antipodal vertices of G, $n = 4$

(ii) any two adjacent vertices or two antipodal vertices of G, when $n \geq 5$.

Theorem 1.5 [3] Let G be a wheel $W_n = K_1 + C_{n-1}(n \geq 4)$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if S consists of

(i) any two adjacent vertices or two antipodal vertices of C_{n-1}, when $4 \leq n \leq 9$.

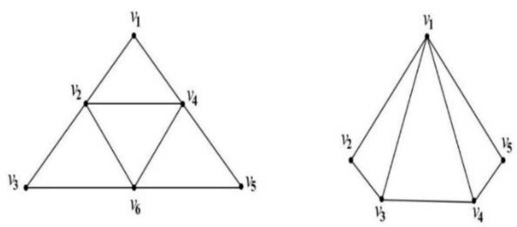
(ii) the vertex of K_1 with any two adjacent vertices of C_{n-1}, when $n \geq 10$ and n is even.

(iii) the vertex of K_1 with any two antipodal vertices of C_{n-1}, when $n \geq 11$ and n is odd.

2. Connected square free detour Number of a graph

Definition 2.1 Let $G = (V, E)$ be a connected graph. A set $S \subseteq V$ is called a connected square free detour set of G if S is the detour set of G and the subgraph $G[S]$ induced by S is connected. The connected square free detour number $cdn_{\square f}(G)$ of G is the minimum order of its connected square free detour sets and any connected square free detour set of order $cdn_{\square f}(G)$ is called a connected square free detour basis of G.

Example 2.2 For the graph G_1 given in Figure 2.1, the subsets $S_1 = \{v_1, v_6\}$, $S_2 = \{v_2, v_5\}$ and $S_3 = \{v_3, v_4\}$ are the three square free detour bases of G_1 so that $dn_{\square f}(G_1) = 2$. It is clear that no two element subset of V is a connected square free detour set of G.



G_1 G_2

Figure 2.1: G_1 and G_2

However, the set $S_4 = \{v_2, v_4, v_6\}$ is the connected square free detour basis of G_1 so that $cdn_{\square f}(G_1) = 3$.

Similarly, for the graph G_2 given in Figure 2.1, the subset $S_5 = \{v_2, v_5\}$ is a square free detour basis but not a connected square free detour set of G_2. The sets $S_6 = \{v_1, v_2, v_4\}$, $S_7 = \{v_1, v_3, v_4\}$, $S_8 = \{v_1, v_3, v_5\}$, $S_9 = \{v_2, v_3, v_4\}$ and $S_{10} = \{v_3, v_4, v_5\}$ are the connected square free detour bases of G_2 that yield $cdn_{\square f}(G_2) = 3$. Thus, there can be more than one connected square free detour basis for a graph G.

Theorem 2.3 For any connected graph G of order $n \geq 2$, $2 \leq dn_{\square f}(G) \leq cdn_{\square f}(G) \leq n$.

Proof. A square free detour set needs at least two vertices so that $dn_{\square f}(G) \geq 2$. Since the connected square free detour set is also a square free detour set, $dn_{\square f}(G) \leq cdn_{\square f}(G)$. Also since G is connected the set of all vertices of G is a connected square free detour set of G, it follows that $dn_{\square f}(G) \leq n$. Thus $2 \leq dn_{\square f}(G) \leq cdn_{\square f}(G) \leq n$.

Corollary 2.4 For any connected graph G, if $cdn_{\square f}(G) = 2$, then $dn_f(G) = 2$.

Proof. This follows from Theorem 2.3.

Definition 2.5 A vertex v in a graph G is a connected square free detour vertex if v belongs to every connected square free detour basis of G. If G has a unique connected square free detour basis of S, then every vertex in S is a connected square free detour vertex of G.

Example 2.6 For the graph G_1 given in Figure 2.1, the subset $S_4 = \{v_2, v_4, v_6\}$ is the unique connected square free detour basis of G_1 so that every vertex of G_1 is a connected square free detour vertex of G_1.

Theorem 2.7 Every end-vertex of a non-trivial connected graph G belongs to every connected square free detour set of G.

Proof. Since every connected square free detour set is also a square free detour set of G the result follows from Theorem 1.1.

Theorem 2.8 Let G be a connected graph with cut-vertices and S a connected square free detour set of G. Then for any cut-vertex y of G, every component of $G - y$ contains an element of S.

Proof: Let y be any cut-vertex of G such that one of the components, say B of $G - y$ contains no element of S. Let $x \in V(B)$. Since S is a connected square free detour set, there exist vertices $u, v \in S$ such that x lies on some $u - v$ square free detour path $P: u = x_0, x_1, x_2, \dots, x, \dots, x_r = v$ in G. Let P' be the $u - x$ subpath of P and P'' be the $x - v$ subpath of P. Since y is a cut vertex of G, both P' and P'' of P contain y, so that P is not a square free detour, which is a contradiction. Thus every component of $G - y$ contains an element of S.

Corollary 2.9 Let G be a connected graph with cut-vertices and S a connected square free detour set of G. Then every branch of G contains an element of S.

Theorem 2.10 Let G be a connected graph with cut-vertices. Then every cut-vertex of G belongs to every connected square free detour set of G.

Proof. Let G be a connected graph and y be a cut-vertex of G. Let $G_1, G_2, G_3, \dots, G_r (r \geq 2)$ be the components of $G - y$. Let S be a connected square free detour set of G. Then by Theorem 2.8, S contains at least one element from each component $G_i (1 \leq i \leq r)$ of $G - y$. Since $G[S]$ is connected, it follows that $y \in S$.

Corollary 2.11 All the end-vertices and cut-vertices of a connected graph G belong to every connected square free detour set of G.

Proof. This follows from Theorems 2.7 and 2.9.

Corollary 2.12 If G is a connected graph of order n such that every vertex x of G is either an end-vertex or a cut-vertex, then $cdn_{\square f}(G) = n$.

Proof. This follows from Corollary 2.11

Remark 2.13 The converse of the corollary 2.12 is not true. For the graph given in Figure 2.2, $cdn_{\square f}(G) = n$, but the vertex x in G is neither a cut-vertex nor an end-vertex.

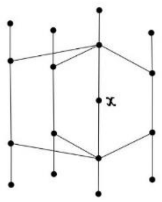


Figure 2.2: G

Corollary 2.14 If T is a tree of order n, then $cdn_{\square f}(T) = n$.

Proof. This follows from Corollary 2.11.

Theorem 2.15 Let $G = \left(K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_r} \cup kK_1 \right) + v$ be a block graph of order $n \geq 4$, such that $r \geq 1$, each $n_i \geq 2$ and $n_1 + n_2 + \cdots + n_r + k =$ n– 1. Then $cdn_{\square f}(G) = r + k + 1$.

Proof. Let $x_1, x_2, x_3, \ldots, x_k$ be the end-vertices of G. Let S be a connected square free detour set of G. Then by Corollary 2.11, $y \in S$ and $x_i \in S (1 \leq i \leq k)$. Also, by Theorem 2.10, S contains a vertex from each component $K_{n_i} (1 \leq i \leq r)$. Now, choose exactly one vertex y_i from each K_{n_i} such that $y_i \in S (1 \leq i \leq r)$. Then $|S| \geq r + k + 1$. Let $S^* = \{y, y_1, y_2, y_3, \ldots, y_r, x_1, x_2, x_3, \ldots, x_k\}$. Since every vertex of G lies on a detour joining a pair of vertices of S^*, it follows that S^* is a square free detour basis of G. Also, since $G[S^*]$ is connected, $cdn_{\square f}(G) = r + k + 1$.

Remark 2.16 If the blocks of the graph G in Theorem 2.15 are not complete, then the theorem is not true.

Theorem 2.17 Let $G = (V, E)$ be a complete graph $K_n (n \geq 2)$ or a cycle $C_n (n \geq 3)$. Then a set $S \subseteq V$ is a connected square free detour basis of G if and only if S consists of any two adjacent vertices of G.

Proof. Let $G = K_n$ be a complete graph of order $n (n \geq 2)$ or a cycle C_n and a set $S \subseteq V$ be a set of two adjacent vertices of G. Then by Theorems 1.2, 1.3 and 1.4, S is a square free detour basis of G. Since any set of two adjacent vertices is connected, it follows that S is a connected square free detour basis of G.

Now, let S be a connected square free detour basis of G. Let S' be any set consisting of two adjacent vertices of G. Then as in the first part of this theorem S' is a connected square free detour basis of G. Hence $|S| = |S'| = 2$ and it follows that S consists of any two adjacent vertices of G.

Theorem 2.18 Let $G = (V, E)$ be a complete bipartite graph $K_{m,n} (2 \leq m \leq n)$ with partitions X and Y where $|X| = m, |Y| = n$. Then a set $S \subseteq V$ is a connected square free detour basis of G if and only if S consists of m vertices of X and exactly one vertex of Y.

Proof. Let $G = K_{m,n} (2 \leq m \leq n)$ be a complete bipartite graph with bipartite sets X and Y. Let $X = \{x_1, x_2, x_3, \ldots, x_m\}$ and $Y = \{y_1, y_2, y_3, \ldots, y_n\}$. Let S^* be a set of m elements of X. Then by Theorem 2.1.20, every vertex of V lies on any square free detour $x_i\, y_j\, x_k (1 \le i \le k \le m,\ 1 \le j \le n, k \ne i)$ of length 2. Thus S^* is a square free detour basis of G. Since the bipartite set X consists of m non-adjacent vertices, the set S^* is not connected. It is clear that $S = S^* \cup \{y : y \in Y\}$ is a connected square free detour set of G. Thus $|S| = m + 1$, consequently S is a connected square free detour basis of G.

Conversely, assume that S is a connected square free detour basis of G. Let S' be any set consisting of $m + 1$ vertices of G. Then as in the first part of this theorem S' is a square free detour basis of G. Hence $|S| = |S'| = m + 1$, it follows that S consists of m vertices of X and any vertex of Y.

Theorem 2.19 Let $G = (V, E)$ be a wheel $W_n = K_1 + C_{n-1} (n \ge 4)$. Then a set $S \subseteq V$ is a connected square free detour basis of G if and only if S consists of

(i) any two adjacent vertices of C_{n-1}, when $n = 4$

(ii) the vertex of K_1 with any two antipodal vertices of C_{n-1}, when $n = 5$

(iii) the vertex of K_1 with any two adjacent vertices of C_{n-1}, when $n \ge 6$.

Proof. This follows from Theorems 1.5 and 2.17.

Corollary 2.20 Let $G = (V, E)$ be a connected graph of order n.

(a) If G is the path P_n or the tree of order n, then $cdn_{\square f}(G) = n$.
(b) If G is the cycle graph C_n, then $cdn_{\square f}(G) = 2$.
(c) If G is the complete graph K_n $(n \ge 2)$, then $cdn_{\square f}(G) = 2$.
(d) If G is the complete bipartite graph $K_{m,n}$ $(2 \le m \le n)$, then $cdn_{\square f}(G) = m + 1$.
(e) If G is the wheel $W_n (n \ge 4)$, then
$$cdn_{\square f}(G) = \begin{cases} 2, & n = 4 \\ 3, & n \ge 5 \end{cases}$$

Proof. (a) This follows from Corollaries 2.12 and 2.14.
(b) This follows from Theorem 2.17.
(c) This follows from Theorem 2.17.
(d) This follows from Theorem 2.18.
(e) This follows from Theorem 2.19.

Theorem 2.21 For each positive integer $k \ge 2$, there exists a connected graph G and a vertex v of degree k in G such that v belongs to every connected square free detour basis of G and $cdn_{\square f}(G) = k$.

Proof. For $k \ge 2$, let $G = K_2 + v$. Then $\deg(v) = 2 = k$, $cdn_{\square f}(G) = 2 = k$ by Corollary 2.17 and the vertex v belongs to every connected square free detour basis of G. For $k \ge 3$, let $G = (K_2 \cup (k-2)K_1 + v)$. Then clearly $\deg(v) = k$ and by Theorem 2.10, the vertex v belongs to every connected square free detour basis of G. Also, by Theorem 2.15, $cdn_{\square f}(G) = 1 + k - 2 + 1 = k$.

Corollary 2.22 For each positive integer $k \ge 2$, and $D_{\square f}$ is square free detour diameter if $G = (K_3 \cup kK_1) + v$ is a block graph, then $cdn_{\square f}(G) = n - D_{\square f} + 1$.

Proof. This follows from Theorems 2.15 and 2.21.

Theorem 2.23 For any three integers α, β and n with $2 \le \alpha < \beta \le n$, there exists a connected graph G with order n and $dn_{\square f}(G) = \alpha$ and $cdn_{\square f}(G) = \beta$.

Proof. We prove this in two cases.

Case 1: If $2 = \alpha < \beta$, let H be a path $P_{n-\alpha+1}$ of order $n - \alpha + 1$ and let G be the graph obtained from H by adding $\alpha - 2$ new vertices to H and joining them to any cut vertex of $P_{n-\alpha+1}$. Then G is the tree of order n and so by Corollary 2.1.9, $dn_{\square f}(G) = \alpha$ and by Corollary 2.14, $cdn_{\square f}(G) = n = \beta$.

Case 2: $2 \leq \alpha < \beta < n$. Let $C = w_1, w_2, w_3, \dots, w_{n-\beta+1}, w_1$ be a cycle of order $n - \beta + 1$ and let $P = v_1, v_2, v_3, \dots, v_{\beta-\alpha+1}$ be a path of order $\beta - \alpha + 1$. Let G_1 be the graph obtained from C and P by identifying w_1 of C with v_1 of P. Let G be the graph obtained from G_1 by adding $\alpha - 2$ new vertices $u_1, u_2, u_3, \dots, u_{\alpha-1}$ to G_1 and joining u_1 to the vertices $w_{n-\beta+1}$ and $w_{n-\beta}$ of C and joining each $u_i (2 \leq i \leq \alpha - 1)$ to the vertex $v_{\beta-\alpha}$ of P. Then G is the connected of order n and shown in Figure 2.3.

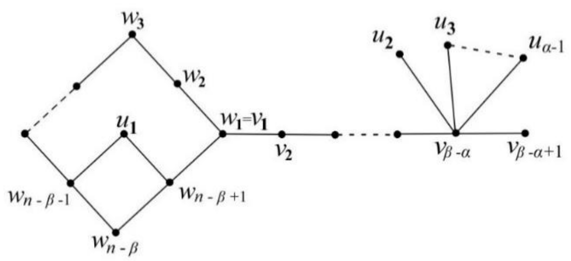


Figure 2.3: G

Let $S_1 = \{u_2, u_3, \dots, u_{\alpha-1}, v_{\beta-\alpha+1}\}$ be the set of all end-vertices of G and let $S_2 = \{v_1, v_2, v_3, \dots, v_{\beta-\alpha}\}$ be the set of all cut-vertices of G. Now, we show that $dn_{\square f}(G) = \alpha$ and $cdn_{\square f}(G) = \beta$. Clearly, the set S_1 is not a square free detour set of G so that $dn_{\square f}(G) \geq |S_1| + 1 = \alpha$. Let $S' = S_1 \cup \{w_2\}$. Since every vertex of G lies on a detour joining a pair of vertices of S', S' is a square free detour set of G and so it follows that from Theorem 1.1 that S' is a square free detour basis of G. Hence $dn_{\square f}(G) = \alpha$. By Corollary 2.13, every connected square free detour set contains $S_1 \cup S_2$. Since $S_1 \cup S_2$ is not a square free detour set of G and since $S = S_1 \cup S_2 \cup \{w_2\}$ is a connected square free detour set of G, it follows that $cdn_{\square f}(G) = |S| = \beta$.

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