MONOPHONIC PEBBLING ON ZERO DIVISOR GRAPHS

ABSTRACT

Assume G is a graph with some pebbles distributed over its vertices. A pebbling move is when two pebbles are removed from one vertex, one is thrown away, and the other is moved to an adjacent vertex. The monophonic pebbling number, $f(G)$, of a connected graph G, is the least positive integer n such that any distribution of n pebbles on G allows one pebble to be carried to any specified but arbitrary vertex using monophonic path by a sequence of pebbling operations. In this paper we find the monophonic pebbling number of some zero divisor graphs.

Keywords: *monophonic pebbling number, monophonic distance, monophonic path.*

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1. Introduction

Pebbling, introduced by Lagarias and Saks, has sparked a lot of interest. F. R. K. Chung [1] was the first to put it into the literature, and many others have followed suit, including Hulbert, who published an overview of graph pebbling [2]. A lot has happened since Hulbert's survey first appeared in graph pebbling. Graph pebbling has been an important instrument for the conveyance of consumable resources for the past 30 years. Assume $G = (V, E)$ be a simple connected graph. Santhakumaran, A. P et al. introduced the monophonic distance in graphs [5]. Lourdusamy et al. [7] defined the monophonic pebbling number of a connected graphs and they find the monophonic pebbling number for various graphs. The line segment that connects two points on a curve is known as a chord. A $u - v$ path is monophonic if it has no chords for any two vertices, u and v, in a connected graph G [5]. The monophonic distance between u and v is the length of the longest $u - v$ monophonic path, notated as $d_m(u, v)$, in G. The monophonic pebbling number of zero divisor graphs are determined in this study.

2. Preliminaries

For graph-theoretic terminology, the reader can go through [4].

Definition 2.1. [6] A chord in a path is an edge joining two non-adjacent vertices of a path. A $u - v$ path with no chords is referred to as a monophonic path. The monophonic pebbling number of a vertex v in G, $\mu(G, v)$, is the least positive integer n such that any distribution of n pebbles on G allows one pebble to be carried to v using monophonic path by a sequence of pebbling moves. The monophonic pebbling number of a graph G, $\mu(G)$, is $\mu(G)$ = $\max_{v \epsilon V}$ $\mu(G, v)$.

Definition 2.2. [6] The zero-divisor graph of a ring R is a simple graph whose set of vertices consists of all (non-zero) zero-divisors, with an edge defined between x and y if and only if xy = 0. It will be denoted by $\Gamma(Z)$.

Note that 2, 3, 4 in Z_6 are zero-divisors. For the element 2 in Z_6 we use y_2, for the element 3 in Z_6 we use y_3 and for the element 4 in Z_6 we use y_4. In general, for the element i in Z_n we use y_i.

Definition 2.3. [3] A complete bipartite graph is a simple bipartite graph with bipartition (V_1, V_2) in which each vertex of V_1 is joined to each vertex of V_2. If $|V_1| = m$ and $|V_2| = n$ then a complete bipartite graph with bipartition (V_1, V_2) is denoted by $K_{m,n}$.

Notation 2.1. The number of pebbles on the vertex v is denoted by $p(v)$. The number of pebbles on the vertex v that is not on the monophonic path is denoted by $p^{\sim}(v)$.

Let $S \subseteq V(G)$. The total number of pebbles placed on the vertices not in S is denoted $p^{\sim}(S)$.

We will use M_i, where $1 \le i \le n$, to denote the monophonic path. we use $M_i^{\sim}$ for the monophonic path which is left after defining M_i. Throughout the paper, we use z to denote the target vertex.

Remark 2.1. Consider the graph G, which has a pebble configuration on its vertices. From G, we select a target vertex z. We can easily shift a pebble to z if $p(z) = 1$ or $p(s) \geq 2$, where $zs \in E(G)$. When z is the target vertex, we always assume that $p(z) = 0$ and $P(s) \leq 1$ for all $zs \in E(G)$.

Result 2.1. Let G be a connected graph. The monophonic distance between u and v is 0 if and only if $u = v$ and 1 when $u - v$ is an edge of G.

Theorem 2.1. [7] For the path P_n, $\mu(P_n)$ is 2^{n-1}.

Theorem 2.2. The monophonic pebbling number for the n-star graph where $n \geq 2$, $\mu(K_{1,n})$ is $n + 2$. We observe that the monophonic distance is equal to the geodesic distance for star graphs. Hence, $f(K_{1,n}) = \mu(K_{1,n}) = n + 2$.

Theorem 2.3. The monophonic pebbling number for the complete bipartite graph is $m + n$. We observe that the monophonic distance is equal to the geodesic distance for complete bipartite graphs. Hence, $f(K_{m,n}) = \mu(K_{m,n}) = m + n$.

Theorem 2.4. The pebbling number of $\Gamma(Z_{16})$ *is* $f(\Gamma(Z_{16})) = 8$.

3. The Monophonic pebbling number of some zero divisor graphs

In this section, we determine the monophonic pebbling number of zero divisor graphs.

Theorem 3.1. For $\Gamma(Z_6)$ *is* $\mu(\Gamma(Z_6)) = 4$.

Proof. Let $V(\Gamma(Z_6))$ be $\{w_2, w_3, w_4\}$ and $E(\Gamma(Z_6))$be$\{(w_2, w_3), (w_3, w_4)\}$. Since $\Gamma(Z_6) \cong P_3$, the result follows from Theorem 2.1.

Theorem 3.2. For $\Gamma(Z_6)$ *is* $\mu(\Gamma(Z_6)) = 4$.

Proof. Let $V(\Gamma(Z_8)) = \{w_2, w_4, w_6\}$ and $E(\Gamma(Z_8)) = \{(w_2, w_4), (w_4, w_6)\}$. Since $\Gamma(Z_8) \cong P_3$, then the result follows by Theorem 2.1.

Theorem 3.3. For $\Gamma(Z_9)$ *is* $\mu(\Gamma(Z_9)) = 2$.

Proof. Let $V(\Gamma(Z_9))$ be $\{w_3, w_6\}$ and $E(\Gamma(Z_9))$ be $\{(w_3, w_6)\}$. This is isomorphic to P_2. Hence, the result follows from Theorem 2.1.

Theorem 3.4. For $\Gamma(Z_{10})$ *is* $\mu(\Gamma(Z_{10})) = 6$.

Proof. Let $V(\Gamma(Z_{10}))$ be $\{w_2, w_4, w_5, w_6, w_8\}$ and $E(\Gamma(Z_{10}))$be$\{(w_2, w_5), (w_4, w_5), (w_6, w_5), (w_8, w_5)\}$. Since $\Gamma(Z_{10}) \cong K_{1,4}$, by Theorem 2.2, $\mu(\Gamma(Z_{10})) = 6$.

Theorem 3.5. For $\Gamma(Z_{12}), \mu(\Gamma(Z_{12})) = 10$.

Proof. Let $V(\Gamma(Z_{12})) = \{w_2, w_3, w_4, w_6, w_8, w_9, w_{10}\}$ and $E(\Gamma(Z_{12})) = \{(w_2, w_6), (w_6, w_8), (w_6, w_4), (w_6, w_{10}), (w_8, w_9), (w_4, w_9), (w_4, w_3), (w_8, w_3)\}$. Place a pebble each on w_{10} and w_3 and 7 pebbles on w_9, we cannot move a pebble to w_2 using the monophonic path. Hence, $\mu(\Gamma(Z_{12})) \geq 10$. Let us consider the distribution of 10 pebbles on$\Gamma(Z_{12})$.

	w_2	w_3	w_4	w_6	w_8	w_9	w_{10}	$d_m(w_i, w_j)$
w_2	0	3	2	1	2	3	2	3
w_3	3	0	1	2	1	2	3	3
w_4	2	1	0	1	2	1	2	2
w_6	1	2	1	0	1	2	1	2
w_8	2	1	2	1	0	1	2	2
w_9	3	2	1	2	1	0	3	3
w_{10}	2	3	2	1	2	3	0	3

Table 1. Monophonic distance of all the vertices $\Gamma(Z_{12})$

Case 1: Let $z = w_j$ where $j = 2,3,9,10$.

Fix $z = w_2$. The monophonic distance from w_2 to any other vertices is ≤ 3. Let the monophonic path M_1 be $\{w_9, w_4, w_6, w_2\}$. Then $\tilde{M_1}$ has the vertices w_3, w_6, w_{10} which are not on M_1. By distributing 8 pebbles on M_1 by Theorem 2.1, we are able to place a pebble on z. If $\mu(z_{12}) - \mu(V(M_1)) \geq 3$, then we can move a pebble to z. If $\mu(z_{12}) - \mu(V(M_1)) \geq 3$ then there will be two possibilities either the pebbling moves take place through the monophonic path M_1 or using the alternative monophonic path $\{w_j, w_8, w_6, w_2\}$. Then we are done.

Case 2: Let $z = w_k$ where $k = 4, 6, 8$.

Fix $z = w_4$. Let the monophonic path M_2 be $\{w_{10}, w_6, w_4\}$ and $V(\tilde{M_2}) = \{w_3, w_8, w_9, w_2\}$. By Theorem 2.1, if $\mu(V(M_2)) \geq 4$, we can reach the target. Otherwise, if $\mu(V(M_2)) < 4$ and $\mu(Z_{12}) - p(V(M_2)) = p(v(\tilde{M_2})) \geq 7$, then we can reach the target. Hence, we are done.

Theorem 3.6. For $\Gamma(Z_{14}), \mu(\Gamma(Z_{14})) = 8$.

Proof. The vertex set of $\Gamma(Z_{14})$ is $\{w_2, w_4, w_6, w_7, w_8, w_{10}, w_{12}\}$ and the edge set of $E(\Gamma(Z_{14}))$ is $\{(w_2, w_7), (w_4, w_7), (w_6, w_7), (w_8, w_7), (w_{10}, w_7), (w_{12}, w_7)\}$. Since $\Gamma(Z_{14}) \cong K_{1,6}$, then by Theorem 2.2, $\mu(\Gamma(Z_{14})) = 8$.

Theorem 3.7. For $\Gamma(Z_{15}), \mu(\Gamma(Z_{15})) = 6$.

Proof. Let the vertex set of $\Gamma(Z_{15})$ be $\{w_3, w_5, w_6 w_9, w_{10}, w_{12}\}$ and the edge set of $\Gamma(Z_{14})$ be $\{(w_3, w_5), (w_9, w_5), (w_{12}, w_5), (w_{10}, w_3), (w_{10}, w_9), (w_{10}, w_{12})(w_6, w_5), (w_6, w_{10})\}$. The monophonic path of $\Gamma(Z_{15})$ is $M{:}\, w_3, w_5, w_9$. Since $\Gamma(Z_{15}) \cong K_{2,4}$, then by Theorem 2.3, $\mu(\Gamma(Z_{15})) = 6$.

Theorem 3.8. For $\Gamma(Z_{16}), \mu(\Gamma(Z_{16})) = 8$.

Proof. Let the vertex set of $\Gamma(Z_{16})$ be $V(\Gamma(Z_{16})) = \{w_2, w_4, w_6, w_8, w_{10}, w_{12}, w_{14}\}$ and the edge set of $\Gamma(Z_{16})$ be $E(\Gamma(Z_{16})) = \{(w_8, w_{12}), (w_8, w_4), (w_8, w_6), (w_8, w_{10}), (w_8, w_{12}), (w_8, w_{14}), (w_4, w_{12})\}$. We observe that for the graph $\Gamma(Z_{16})$ the monophonic distance is equal to the geodesic distance. Thus, by Theorem 2.4, $\mu(\Gamma(Z_{16})) = 8$.

Theorem 3.9. For $\Gamma(Z_{18})$, $\mu(\Gamma(Z_{18})) = 14$.

Proof. Let the vertex set of $\Gamma(Z_{18})$ be $\{w_2, w_3, w_4, w_6, w_8, w_9, w_{10}, w_{12}, w_{14}, w_{15}, w_{16}\}$ and the edge set of $\Gamma(Z_{18})$ be $\{w_9 w_i, w_6 w_j, w_{12} w_{15}, w_{12} w_3\}$ where $i = 6, 12, 2, 4, 8, 10, 14, 16$ and $j = 12, 15$. To prove the necessary part, let $z = w_{16}$. Without loss of generality, consider the monophonic path $M: w_{16}, w_9, w_{12}, w_{15}$. Place a pebble each on $w_{14}, w_{13}, w_{10}, w_8, w_4, w_2$ and 7 pebbles on w_{16}. Then we cannot move a pebble to z using the monophonic path. Hence, $\mu(\Gamma(Z_{18})) \geq 14$. Let us consider the distribution of 14 pebbles on $\Gamma(Z_{18})$.

	w_2	w_3	w_4	w_6	w_8	w_9	w_{10}	w_{12}	w_{14}	w_{15}	w_{16}	$d_m(w_i, w_j)$
w_2	0	3	2	2	2	1	2	2	2	3	2	3
w_3	3	0	3	1	3	2	3	1	3	2	3	3
w_4	2	3	0	2	2	1	2	2	2	3	2	3
w_6	2	1	2	0	2	1	2	1	2	1	2	2
w_8	2	3	2	2	0	1	2	2	2	3	2	3
w_9	1	2	1	1	1	0	1	1	1	2	1	2
w_{10}	2	3	2	2	2	1	0	2	2	3	2	3
w_{12}	2	1	2	1	2	1	2	0	2	1	1	2
w_{14}	2	3	2	2	2	1	2	2	0	3	2	3
w_{15}	3	2	3	1	3	2	3	1	3	0	3	3
w_{16}	2	3	2	2	2	1	2	2	2	3	0	3

Table 2. Monophonic distance between all pairs of vertices $\Gamma(Z_{18})$

Case 1: Let $z = w_j$ where $j = \{2, 4, 8, 10, 13, 14, 15, 16\}$.

Fix $z = w_2$. Then $d_m(w_2, w_x) \leq 3$ where $w_x \in V(\Gamma(Z_{18}))$. Let us consider the monophonic path $M_1: w_{13}, w_{12}, w_9, w_2$. If $p(V(P_1)) \geq 8$, we are done by Theorem 2.1. Suppose $p(V(M_1)) < 8$. Let $V(\tilde{M_1}) = \{w_{15}, w_6, w_4, w_8, w_{10}, w_{14}, w_{16}\}$. We can reach the target for the following conditions. If $\frac{p(w_{15})}{2} + p(w_6) \geq 4$ or $\frac{p(N(w_9))}{2} + p(w_9) \geq 2$, we are done.

Case 2: Let $z = w_k$ where $k = \{13, 15\}$.

Fix $z = w_{13}$. Let us consider the monophonic path $M_2: w_{13}, w_{12}, w_9, w_2$ and $V(\tilde{M_2}) = \{w_{15}, w_6, w_4, w_8, w_{10}, w_{14}, w_{16}\}$. If $p(V(M_2)) \geq 8$, we are done by Theorem 2.1. Suppose $p(V(M_2)) < 8$, then $\frac{p(w_{15})}{2} + p(w_{12}) \geq 2$ or $\frac{p(V\tilde{M_2})}{2} + p(w_9) \geq 4$, we can reach the target.

Case 3: Let $z = w_9$.

Without loss of generality, let $M_3: w_{15}, w_6, w_9$ and $V(\tilde{M_3}) = \{w_{13}, w_{12}, w_2, w_4, w_8, w_{10}, w_{14}, w_{16}\}$. If $p(V(M_3)) \geq 4$ we can reach the target by Theorem 2.1, without using the pebbles from $\tilde{M_3}$. Suppose $p(V(M_3)) < 4$. If any one of the vertices of $N(w_9)$ has at least 2 pebbles or $\left\lfloor \frac{p(w_{13})}{2} \right\rfloor \geq 2$, we are done.

Case 4: Let $z = w_s$ where $s = \{w_6, w_{12}\}$.

Without loss of generality, let $z = w_6$. Let $M_4: w_6, w_9, w_2$. Since w_9 is the neighbourhood of w_k, w_{12}, w_6 where $k = 2, 4, 8, 10, 14, 16$. If w_9 receives at least 2 pebbles after the pebbling moves from w_k we are done.

Thus, $\mu(\Gamma(Z_{18})) = 14$.

Theorem 3.10. For $\Gamma(Z_{2p})$, $\mu(\Gamma(Z_{2p})) = p + 1$, where p is any prime number.

Proof. Let the vertex set of $\Gamma(Z_{2p})$ be $V(\Gamma(Z_{2p})) = \{w_2, w_4, \cdots, w_{2p-2}, w_p\}$ and the edge set of $\Gamma(Z_{2p})$ be $E(\Gamma(Z_{2p})) = \{w_i w_p\}$ where $2 \leq i \leq 2p - 2$. Since $\Gamma(Z_{2p}) \cong K_{1,p-1}$, by Theorem 2.2, we can move a pebble to any vertex of $\Gamma(Z_{2p})$.

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