d-LUCKY NUMBER OF GRAPHS FROM COMMUTATIVE RING

ABSTRACT

A d-lucky labeling $l: V \to \{1,2,\dots,k\}$ of a graph $G = (V, E)$ is a labeling of vertices in such a way that any two different incident vertices u and v, their colors $c(u) = d(u) + \sum_{v\in N(u)} l(v)$, $c(v) = d(v) + \sum_{u\in N(v)} l(u)$ are distinct where $d(u)$ denotes the degree of u in a graph and $N(u)$ denotes the open neighbourhood of u in a graph. In this paper we determine the d-lucky number for some zero divisor graphs.

Keyword: *d-lucky labeling, d-lucky number, zero divisor graphs*

Introduction:

An assignment of colors to the vertices of a graph so that no two incident vertices get the same color is called a coloring of the graph. The graph coloring has many real life applications like scheduling flights to specific routes, assigning frequency channels to different wireless applications, scheduling exams etc. In [1], Czerwinski, Grytczuk, Zelezny introduced the concept of lucky labeling. In [2], Mirka Miller, Indira Rajasingh, D.Ahima Emilet, D.Azhubha Jemilet, introduced the concept of d-lucky labeling. Let $l: V(G) \to N$ is a vertex labelling, where N denotes the set of all natural numbers. The labeling l is a d-lucky labeling if $c(u) \neq c(v)$ holds for every pair of incident vertices of u and v, where

$$c(u) = \mathrm{d(u)} + \sum_{\mathrm{v}\in \mathrm{N(u)}} l(\mathrm{v})$$

$$c(v) = \mathrm{d(u)} + \sum_{\mathrm{u}\in \mathrm{N(v)}} l(\mathrm{u})$$

d(u) is the degree of u and N(u) be the open neighbourhood of the vertex u in a graph. The d-lucky number of a graph is the minimum value of the labeling needed to label the graph.

In this paper, we mainly focused on the d-lucky number of some zero divisor graphs. First, the idea of zero divisor graph introduced by I.Beck. Later Anderson and Livingston [3] modified the concept of zero divisor graph. I.Beck [4] considered all the zero divisors of a commutative ring as the vertices of the graph and connect distinct vertices if the product of the distinct vertices is equal to zero. Anderson and Livingston considered all the non zero zero divisor as the vertices of the graph and join the distinct vertices if the product of the distinct vertices is equal to zero. Consider R be a commutative ring and Z(R) be its set of all zero divisors of R, the zero divisor graph denoted by Γ(R), the vertices of zero

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divisor graph, $Z(R)^* = Z(R) \setminus \{0\}$, the set of all non zero zero divisors of R and the distinct vertices of the this graph are adjacent if their product gives zero.

d-lucky labeling of some zero divisor graphs: :

In this section, we determine the d-lucky number of some zero divisor graphs.

Definition 2.0: [3] A d-lucky labeling $l: \mathrm{V} \rightarrow \{1,2,\dots,\mathrm{k}\}$ of a graph $\mathrm{G} = (\mathrm{V},\mathrm{E})$ is a labeling of vertices in such a way that for any two different incident vertices u and v, their colors $\mathrm{c(u)} = \sum_{\mathrm{v}\in \mathrm{N(u)}} l(\mathrm{v}) + \mathrm{d(u)}$ and $\mathrm{c(v)} = \sum_{\mathrm{u}\in \mathrm{N(v)}} l(\mathrm{u}) + \mathrm{d(v)}$ are distinct. Where d(u) denotes the degree of u and N(u) denotes the open neighbourhood of u. The d-lucky number of G, $\eta_{dl}(\mathrm{G})$ is defined as the minimum k for which the graph G has a d-lucky labeling.

Theorem 2.1: Let n be an odd prime number. For a zero divisor graph $\Gamma(R)$, the d-lucky number, $\eta_{dl}((\Gamma(\mathrm{R})) = 1$ where $R = \mathbb{Z}_{2^x n}, x = 1, 2$.

Proof: Let $\Gamma(R)$ be a zero divisor graph where $R = \mathbb{Z}_{2^x n}$.

Case (i): Suppose $x = 1$ and let n be an odd prime. By the definition of zero divisor graph, Let us consider the vertex set $V(\Gamma(R)) = \{2,4,..2(n-1),n\} = \{v_i : 1 \leq i \leq n\}$, and edge set $E(\Gamma(R)) = \left\{v_i\, v_n : v_i \in V(\Gamma(R))\setminus\{v_n\}\right\}$. Then we have the vertex degrees $d(v_i) = m-1,\ 1 \leq i \leq 2(n-1)$ and $d(v_n) = n-1$. Define $l: V(\Gamma(R)) \rightarrow \{1,2,\dots,k\}$ as follows: $l(v_i) = 1;\ 1 \leq i \leq n$. We observe that, $c(v_i) = 2m-2,\ c(v_n) = 2n-2$. Therefore $c(v_i) \neq c(v_n),\ 1 \leq i \leq n$. Hence $\eta_{dl}((\Gamma(\mathrm{R})) = 1$.

Case (ii): Suppose $x = 2$ and $V(\Gamma(R))$ has partitioned into two sets $V_1(\Gamma(R))$ and $V_2(\Gamma(R))$. Now $V_1(\Gamma(R))$ contains the multiples of n in $\mathbb{Z}_{2^x n}$, $V_2(\Gamma(R))$ contains the multiples of m excluding $2n$ in $\mathbb{Z}_{m^x n}$. Let $V_1(\Gamma(R)) = \{r_1, r_2, r_3\}$ and $V_2(\Gamma(R))$=$\{s_1, s_2, \dots, s_{n-1}, s_{n+1}, \dots, s_{2n-1}\}$ such that $|V(\Gamma(R))| = 2n+1$ and $E(\Gamma(R)) = \left\{r_i s_j : i \in \{1,3\},\ s_j = \{4,8,\dots,(4m-4)\}\right\} \cup \left\{r_2 s_j : s_j \in V_2(\Gamma(R))\right\}$ and $|E(\Gamma(R))| = 4n-4$. Hence $d(r_i) = n-1, for\ i \in \{1,3\}$; $d(r_2) = 2n-2$; $d(s_j) = m+1,\ s_j \in \{4,8,\dots,4n-4\}$; $d(s_j) = m-1, s_j \in V_2(\Gamma(R))\setminus \{4,8,\dots,4n-4\}$.

Define a labeling $l: V(\Gamma(R)) \rightarrow \{1,2,\dots,k\}$ as follows: $l(r_i) = 1\ for\ 1 \leq i \leq 3$; $l(s_j) = 1$ for $1 \leq j \leq 2n-2$. We observe that,
$c(r_i) = 2n-2,\ i \in \{1,3\}$
$c(r_2) = 4n-4$;
$c(s_j) = 2m-2, s_j \in V(\Gamma(R))\setminus \{4, 8, \dots, 4n-4\}$,
$c(s_j) = 2m+2, s_j \in \{4,8,\dots,4n-4\}$.
Thus$c(r_i) \neq c(s_j)$
for all $\mathrm{r_i} \in \mathrm{V_1}(\Gamma(\mathrm{R})), \mathrm{s_j} \in \mathrm{V_2}(\Gamma(\mathrm{R}))$.
Hence $\eta_{dl}((\Gamma(\mathrm{R})) = 1$.
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Theorem 2.2: For a zero divisor graph $\Gamma(R)$, $R = \mathbb{Z}_{3n}, n > 3$ be an odd prime, $\eta_{dl}((\Gamma(\mathrm{R})) = 1$.

Proof: Let $\Gamma(R)$ be a zero divisor graph. Where $R = \mathbb{Z}_{3n}, n > 3$ be an odd prime. In this graph, the vertex $V(\Gamma(R))$ can be partitioned $V_1(\Gamma(R))$ and $V_2(\Gamma(R))$ where $V_1(\Gamma(R)) = \{n, 2n\}$, $V_2(\Gamma(R)) = \{3i: 1 \le i \le n\}$ and $E(\Gamma(R)) = \{uv: u \in V_1(\Gamma(R)), v \in V_2(\Gamma(R))\}$. Hence the degrees of the vertices u and v, $d(u) = n-1, d(v) = m-1$ for all $u \in V_1(\Gamma(R)), v \in V_2(\Gamma(R))$ and $|E(\Gamma(R))| = 2n-2$.

Define a labeling $l\colon V(\Gamma(R)) \to \{1,2,\dots,k\}$ as follows: $l(u) = 1$ for all $u \in V_1(\Gamma(R))$; $l(v) = 1$ for all $v \in V_2(\Gamma(R))$. Note that, $c(u) = 2n-2$, $c(v) = 2m-2$. Then it is clear that $c(u)$ and $c(v)$ are distinct for all $u \in V_1(\Gamma(R)), v \in V_2(\Gamma(R))$. Hence $\eta_{dl}((\Gamma(R)) = 1$.

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Theorem 2.3: Let $m = 3, n > 3$ be a prime. For a zero divisor graph $\Gamma(R), \eta_{dl}((\Gamma(R)) = 2$, where $R = \mathbb{Z}_{m^2 n}$.

Proof: Let R be a commutative ring and $\Gamma(R)$ be a zero divisor graph. In this case, the vertex set of $\Gamma(R)$ can be partitioned into two sets V_1 and V_2 such that $V_1 = \{n, 2n, 3n, 4n, 5n, 6n, 7n, 8n\}$ and $V_2 = \{3,6,9,\dots,9n-3\} \setminus \{n, 2n\} = \{v_1, v_2, \dots, v_{3n-1}\}$. Now the edge set $E(\Gamma(R)) = \{u_i v_i: u_i \in V_1,\ v_i \in \{9,18,27,\dots,9(n-1)\}\} \cup \{u_i v_i : u_i \in \{3n, 6n\}, v_i \in V_2\} \cup \{u_3, u_6\}$. Hence $d(u_i) = n-1$ for all $u_i \in V_1 \backslash \{3n, 6n\}$; and $d(u_i) = 3n-2, i = \{3,6\}; d(v_i) = 8$ for all $v_i \in \{9,18,\dots,9(n-1)\}$ and $d(v_i) = 2, v_i \in V_2\ \{9,18,\dots,9(n-1)\}$.

Define the labeling $l\colon V(\Gamma(R)) \to \{1,2,\dots,k\}$as follows: $l(u_i) = 1, 1 \le i \le 8$ except u_6; and then label u_6 as $l(u_6) = 2$, $u_6 \in V_1; l(v_i) = 1$ for all $v_i \in V_2$. We observe that, $c(u_i) = 2n-2 \ \ 1 \le i \le 8$ except u_3 and u_6; $c(u_3) = 6n-3$; $c(u_6) = 6n-4$; $c(v_i) = 5$ for all $v_i \in V_2 \setminus \{9,18,\dots,9(n-1)\}, c(v_i) = 17$, $v_i \in \{9,18,\dots,9(n-1)\}$. We can be easily verified that $c(u_i) \neq c(v_i)$ for all $u_i \in V_1$, $v_i \in V_2$. Hence $\eta_{dl}((\Gamma(R)) = 2$.

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Theorem 2.4: Let $R = \prod_{i=1}^{k} Z_{m_i^{n_i}}$ be a commutative ring with unity. For a zero divisor graph $\Gamma(R), \eta_{dl}((\Gamma(R)) = M-1$ where $M = Max\,(m_1, m_2, \dots, m_k)$ and m_i's are distinct primes, n_i's are positive integers.

Proof: Consider $\Gamma(R)$ be a zero divisor graph of commutative ring $R = \prod_{i=1}^{k} Z_{m_i^{n_i}}$, m_i's are distinct primes, n_i's are positive integers. The vertex set of $\Gamma(R)$ consists of different blocks , $V(\Gamma(R)) = \cup B_{x_1, x_2, \dots, x_k}$ where $(x_1, x_2, \dots, x_k) \neq (0, 0, \dots, 0)$ and $(x_1, x_2, \dots, x_k) \neq (n_1, n_2, \dots, n_k)$.$B_{x_1, x_2, \dots, x_k} = \{(u_1, u_2, \dots, u_k)$: $u_i = 0$ if $x_i = n_i$ and $m_i^{x_i} \mid u_i$ and $m_i^{x_i+1} \nmid u_i$ if $x_i \in \{0,1,2, \dots, n_i-1\}\}$. All the vertices in $B_{x_1, x_2, \dots, x_k}$ are adjacent to all the vertices in $B_{y_1, y_2, \dots, y_k}$ if $x_i + y_i \ge n_i$ for all $i = 1, 2, \dots, k$.

The vertices in $B_{x_1, x_2, \dots, x_k}$ form a clique in $\Gamma(R)$ if $2x_i \ge n_i$ for all $i = 1, 2, \dots k$. Hence we have, for each $u \in B_{x_1, x_2, \dots, x_k}$, deg u = $-2 + \prod_{i=1}^{k} m_i^{x_i}$ if $B_{x_1, x_2, \dots, x_k}$ is clique; deg u = $-1 + \prod_{i=1}^{k} m_i^{x_i}$ if $B_{x_1, x_2, \dots, x_k}$ is not a clique.

Define a labeling $t : V(\Gamma(R)) \to \{1,2,\dots,k\}$ as follows: Label the vertices of the block as 1 if the block is not form a clique. If the block is form a clique, label the vertices of clique as $1 \le t \le |B_{x_1, x_2, \dots, x_k}|$ where $|B_{x_1, x_2, \dots, x_k}| = \prod_{i=1}^{k} \varphi(m_i^{n_i - x_i})$, Let T = Max

$(|B_{x_1,x_2,\ldots,x_k}|)$ if $B_{x_1,x_2,\ldots,x_k}$ form a clique

We observe that, for each $u \in B_{x_1,x_2,\ldots,x_k}$,

$c(u) = -1 + \prod_{i=1}^{k} m_i^{x_i} + \sum_{v \in N(u)} l(v)$ if $B_{x_1,x_2,\ldots,x_k}$ is not a clique

$c(u) = -2 + \prod_{i=1}^{k} m_i^{x_i} + \sum_{v \in N(u)} l(v)$ if $B_{x_1,x_2,\ldots,x_k}$ forms a clique.

It is clear that colors of all the incident vertices are pairwise distinct. Hence $\eta_{dl}((\Gamma(R)) = T = M - 1$ where $M = Max\ (m_1, m_2, \ldots, m_k)$.

Conclusion:

In this paper we determined the d-lucky number of some zero divisor graphs. In this context we can extend this result into we can able to find d lucky number for any kind of zero divisor graph as well as other algebraic structured graphs like annihilating ideal graphs, Cayley graph, unitary addition Cayley graph etc.,

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