OPERATION ORDER DIVISOR GRAPH OF A GROUP

ABSTRACT

Consider a finite group $(G,*)$ *. The Operation Order Divisor graph* $\Gamma_{ood}(G)$ *of* G *is a graph with* $V(\Gamma_{ood}(G)) = G$ *and two different vertices* a *and* b *have an edge in* $\Gamma_{ood}(G)$ *if and only if either* $O(a)|O(a*b)$ *or* $O(b)|O(a*b)$*. Here we are going to compare the properties of* G *and the properties of* $\Gamma_{ood}(G)$.

Keywords *: Operation Order Divisor graph, complete graph, finite group.*
MSC 2010: 05C 25

Introduction:

The algebraic graph theory becomes an exciting research topic in the last twenty years.Many researches are developed on getting a graph from a algebra structure and then find out the properties of the algebraic structure using the resulting graph.

The terms and definitions in graph theory are referred from [5] and for that of algebra are referred from [6].

In 2009, Sattanathan and Kala[7] introduced order prime graph of a group. In this graph vertices are the group elements and any two vertices are adjacent if and only if their orders are relatively prime. The concept of an order divisor graph of a group is introduced in [8] and in [9] with a slight change. While [8] defined it on elements of a finite group,[9] defined it on subgroups of a finite group. For [8], the order divisor graph, denoted as $OD(G)$, whose vertex set is G such that two distinct vertices x and y having different orders are adjacent provided that $O(x)|O(y)$ or $O(y)|O(x)$.

Motivated by the these concepts, we define and introduce the concept of operation order divisor graphs.

Operation order divisor graph

Here we discuss about some preliminary properties of operation order divisor graphs.

Proposition 2.1 *Consider a finite group* $(G,*)$ *with order n and identity* e*. Then the degree of* e *in* $\Gamma_{ood}(G)$ *is n-1.*

R M MUTHULAKSHMI,
Research Scholar, Reg No : 19221172092023,
Department of Mathematics,Rani Anna Government College for Women,Affliated to Manonmaniam Sundaranar University
Tiruneveli – 627012,
Mobile No : 9843372602,
E-mail:muthu72602@gmail.com

Dr N LALITHA,
Department of Mathematics,Thiruvalluvar College, Papanasam,Tirunelveli 627425,
Mobile No : 9488063178,
E-mail: lalitha.msu12@gmail.com

Dr D PREMALATHA
Department of Mathematics
Rani Anna Government College for Women
Tirunelveli
Mobile No : 9487751385
E-mail: lathaaedward@gmail.com

Proof. Consider any element $a \in G$.Then obviously $O(e)|O(a*e)$. Therefore e will have adjacency to each and every other n-1 elements in G.Therefore the degree of e in $\Gamma_{ood}(G)$ is n-1.

Proposition 2.2 *Consider a finite group $(G,*)$ and $x \in G$ be not a self inverse element. Then x and x^{-1} does not have an edge in $\Gamma_{ood}(G)$.*

Proof. Let x be a non self inverse element of the group . Clearly $x*x^{-1}=e$, the identity element of G. Therefore $O(x), O(x^{-1})$does not divide $O(e)$ .Thus x and x^{-1} does not have an edge in $\Gamma_{ood}(G)$.

Proposition 2.3 *Consider a finite group $(G,*)$ and an element x such that $O(x) > 3$. Then x and x^{-2} have an edge in $\Gamma_{ood}(G)$.*

Proof. Let $x \in G$ be an element of order greater than 3. Clearly $x*x^{-2}=x^{-1}$ and $O(x)=O(x^{-1})$. Therefore $O(x)$ divides $O(x*x^{-2})$ .Thus x and x^{-2} have an edge in $\Gamma_{ood}(G)$.

Theorem 2.4 *For any finite group $(G,*)$, $\Gamma_{ood}(G)$ is a complete graph if and only if all the elements except identity has self inverse.*

Proof. Consider a finite group $(G,*)$. Assume that $\Gamma_{ood}(G)$ is a complete graph. Suppose there exists an element x in G such that x does not have a self inverse. Proposition 2.2 states that x and x^{-1} does not have an edge, which is a contradiction to our assumption that G is complete. Hence every element other than identity has self inverse in G. Conversely, assume that every element other than identity has a self inverse. Therefore $x*y \neq e$ for all $x \neq y \in G$ and also $O(x)=2$ for all $x \in G-e$. x and y have an edge in $\Gamma_{ood}(G)$.Thus $\Gamma_{ood}(G)$ is a complete graph.

Theorem 2.5 *For any finite group $(G,*)$, $\Gamma_{ood}(G)$ is a star graph if and only if G is isomorphic to one of the groups $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. Clearly $\Gamma_{ood}(\mathbb{Z}_2)=K_2$ and $\Gamma_{ood}(\mathbb{Z}_3)=K_{1,2}$ and hence $\Gamma_{ood}(G)$ is a star graph, when G is either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Conversely, Let us assume that $\Gamma_{ood}(G)$ is a star graph. By proposition2.3, G has no element of order greater than 3. Then every element other than identity has order either 2 or 3. Suppose G has two elements x and y of orders 2 and 3 respectively. Clearly $O(x*y)$ is either 2 or 3. Therefore x and y have adjacency, which contradicts our assumption. Now we have two possible cases.

Case(i): Every element other than identity has order 2

By Theorem 2.4, $\Gamma_{ood}(G)$ is a complete graph. We know that both complete and star graph is K_2 and hence G must be $\mathbb{Z}_2$.

Case(ii): Every element other than identity has order 3

In this case two non identity elements x and y are adjacent if $y \neq x^{-1}$. Therefore $\Gamma_{ood}(G)$ is a star graph if G must be $\mathbb{Z}_3$.

Thus we can conclude that G is either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Since the identity element is an element having full degree, Star graph is the only tree, from the above theorem we have the following corollary.

Corollary 2.6 *For any finite group $(G,*)$, $\Gamma_{ood}(G)$ is a tree if and only if G is isomorphic to one of the groups $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Theorem 2.7 *Let $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p$, where p is an odd prime number and $O(G)=p^n$. Then $\Gamma_{ood}(G) \cong K_{1,2,2,\ldots,2}$.*

Proof. Let $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p$, where p is an odd prime number and $O(G) = p^n$. Consider the identity element as a single partition. Note that every non identity element has order p.Since p is odd , G has no self inverse element. Therefore two non identity elements x and y have an edge in $\Gamma_{ood}(G)$, if $y \neq x^{-1}$. Therefore we divide the remaining non identity elements into partitions such that each partition has 2 elements x and x^{-1}. Hence $\Gamma_{ood}(G) \cong K_{1,2,2,\ldots,2}$

Theorem 2.8 *$\Gamma_{ood}(G)$ can not be a cycle for any finite group $(G,*)$.*

Proof. By Proposition 2.1, the identity element is adjacent to all the other elements in $\Gamma_{ood}(G)$. Therefore $O(G) = 3$ when $\Gamma_{ood}(G)$ is cycle. But note that every group of order 3 must be isomorphic to $\mathbb{Z}_3$. $o(G) \cong K_{1,2}$. Hence $\Gamma_{ood}(G)$ can not be cycle.

Theorem 2.9 *Consider $(G,*)$ as a group having p^2 elements, where p is an odd prime. Then the number of edges of*

$$\Gamma_{ood}(G) = \begin{cases} \frac{(p-1)(p^3-1)}{2} & if G is cyclic \\ \frac{(p^2-1)^2}{2} & if G is noncyclic \end{cases}$$

Proof. Consider $(G,*)$ as a group of order p^2, where p is an odd prime. Since a group of order p^2, where p is prime must be abelian, G is an abelian group. Therefore we have the following cases.

Case 1:*G* is cyclic

Let G be cyclic. $G \cong \mathbb{Z}_{p^2}$. Then we can divide G into three sets A, B, C such that A is a set of all elements having order 1, B is a set of all elements having order p and C is a set of all elements having order p^2. Since G is cyclic, $|A| = 1, |B| = \phi(p) = p - 1, |C| = \phi(p^2) = p(p-1)$. By proposition 2.1, $deg(e) = p^2 - 1$ .Let $x \in B$. x is adjacent to all the elements of B except it inverse. If $y \in C$, then either $x * y \in B$ or $x * y \in C$. Since G has unique subgroup of order p, $x * y \notin B$. Therefore $x * y \in C$ and so x and y are adjacent. Therefore $deg(x) = 1 + (p-3) + p(p-1) = p^2 - 2$ or all $x \in B$. Let $x \in C$. Clearly x is adjacent to all the elements of sets A and B. Let $y \in B$. Note that $x^{-1}, x^{-1} * y \in C$,But $x * x^{-1} * y \in B$. Therefore x is not adjacent to x^{-1} and $x^{-1} * y$ for all $y \in B$. Therefore $deg(x) = p^2 - 1 - p = p^2 - p - 1$ or all $x \in C$. Therefore the sum of the degrees of all elements of G is $(p^2-1) + (p-1)(p^2-2) + (p^2-p)(p^2-p-1) = (p-1)(p^3-1)$. Hence the number of edges of $\Gamma_{ood}(G) = \frac{(p-1)(p^3-1)}{2}$.

Case 2: *G* is non cyclic

Since G is non cyclic abelian group, $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore by Theorem 2.7, $o(G) \cong K_{1,2,2,\ldots,2}$. Therefore the number of edges of $\Gamma_{ood}(G) = \frac{(p^2-1)^2}{2}$.

Theorem 2.10 *Consider $(G,*)$ as a cyclic group of order $2p$, where p is an odd prime number. Then $\Gamma_{ood}(G)$ has $\frac{3p^2-4p+3}{2}$ edges.*

Proof. Consider $(G,*)$ as a cyclic group of order $2p$. Then we can divide G into four sets A, B, C, D such that A is a set of all elements having order 1, B is a set of all elements having order 2, C is a set of all elements having order p, D is a set of all elements having order $2p$. Since G is cyclic,

$|A| = 1, |B| = \phi(2) = 1, |C| = \phi(p) = p - 1$ and $|D| = p - 1$. By proposition 2.1, $deg(e) = 2p - 1$.

Case(i): Consider an element $x \in B$.

x is adjacent to e. If $y \in C$, then $O(x * y) = 2p$. Therefore $x * y \in D$ and so each $x \in B$ is adjacent to all $y \in C$. For every elements $y \in C$, $x * y$ must be in D. Therefore $x * x * y = y \in C$ and so x is not adjacent to any elements in D. Hence $deg(x) = 1 + (p - 1) = p$, for all $x \in B$.

Case(ii): Let $x \in C$.

Clearly x is adjacent to all the elements of sets A and B. In C, x is adjacent to all elements in C expect its inverse. Let $x \in C$. For every elements $y \in B$, $x * y$ must be in D. Therefore $x * x * y = y \in B$ and so x is not adjacent to $x * y$ in D but x is adjacent to remaining elements of D. Hence $deg(x) = 1 + 1 + (p - 3) + (p - 2) = 2p - 3$, for all $x \in C$.

Case(iii): Let $x \in D$.

Clearly x is adjacent to all the elements of sets A and $C - \{x^{-1} * y\}$, where $y \in B$. Note that if $x \in B$ and $y \in D$ then $x * y \in C$. Therefore no element in D is adjacent to element in B. Consider two elements x, y in D. clearly $x = a * b_1$ and $y = a * b_2$, where $a \in B, b_1, b_2 \in C$. $x * y = b_1 * b_2 \in C$ or B. Therefore x and y does not have an edge. Hence $deg(x) = 1 + (p - 2) = p - 1$, for all $x \in D$. Therefore the sum of the degrees of all elements of G is $(2p - 1) + p + (p - 1)(2p - 3) + (p - 1)(p - 1) = 3p^2 - 4p + 3$. Hence $\Gamma_{ood}(G)$ has $\frac{3p^2-4p+3}{2}$ edges.

Theorem 2.11 Consider $(G,*)$ as a cyclic group of order pq, where p, q are distinct odd prime .Then $\Gamma_{ood}(G)$ has $\frac{(p^2-p-1)(q^2-q-1)-pq}{2}$ edges

Proof. Consider $(G,*)$ as a cyclic group of order pq, where p, q are distinct odd primes. Then G can be divided into four sets A, B, C, D such that A is a set of all elements having order 1, B is a set of all elements having order p, C is a set of all elements having order q, D is a set of all elements having order pq. Since G is cyclic, $|A| = 1, |B| = \phi(p) = p - 1, |C| = \phi(q) = q - 1$ and $|D| = pq - p - q + 1$. By proposition 2.1, $deg(e) = pq - 1$.

Case(i): Let $x \in B$.

x is adjacent to e and all elements of B other than its inverse. If $y \in C$, then $O(x * y) = pq$. Therefore $x * y \in D$ and so x is adjacent to all $y \in C$. For every elements $y \in C$, $x^{-1} * y$ must be in D. Therefore $x * x^{-1} * y = y \in C$ and so x is not adjacent to the elements in D is of the form $x^{-1} * y$, where $x \in B$ and $y \in C$. Hence $deg(x) = 1 + (p - 3) + (q - 1) + (pq - p - 2q + 2) = pq - q - 1$, for all $x \in B$.

Case(ii): Let $x \in C$.

we use the argument of case(i), we get $deg(x) = 1 + (p - 1) + (q - 3) + (pq - 2p - q + 2) = pq - p - 1$, for all $x \in C$.

Case(iii:) Let $x \in D$.

Clearly x is adjacent to all the elements of sets A. Note that every element in D is of the form product of one element in B and one element in C. Therefore $x = a * b$, where $a \in B$ and $b \in C$. Clearly x is not adjacent to $a^{-1} \in B$ and adjacent to remaining elements in$B$. Similarly x is not adjacent to $b^{-1} \in C$ and adjacent to remaining elements is C. In D, x is not adjacent to the elements of the form $a^{-1} * b$, for all $b \in C$ and $a * b^{-1}$, for all $a \in B$ and x is adjacent to the remaining elements of D. Hence $deg(x) = 1 + (p - 2) + (q - 2) + (pq - 2p - 2q + 3) = pq - p - q$, for all $x \in D$. Therefore the sum of the degrees of all elements of G is $(pq - 1) + (p - 1)(pq - q - 1) + (q - 1)(pq - p - 1) + (pq - p - q + 1)(pq - p - q) = (p^2 - p - 1)(q^2 - q - 1) - pq$. Hence $\Gamma_{ood}(G)$ has $\frac{(p^2-p-1)(q^2-q-1)-pq}{2}$ edges.

References

1. *A. Abdollahi, S. Akbari, H.R. Maimani. Non-Commuting graph of a group, Journal of Algebra, 2006, 298: 468-492.*
2. *S. Akbari, A. Mohammadian. On the zero-divisor graph of commutative ring, Journal of Algebra, 2004, 274(2): 847-855.*
3. *Tamizh Chelvam, T., and Sattanathan, M., Subgroup intersection graph of a group, J. Adv. Research in Pure Math, 3 no. 4 (2011) 44-49. (doi:10.5373/ jarpm .594.100910, ISSN:1943-2380).*
4. *Tamizh Chelvam, T., and Sattanathan, M., Subgroup intersection graph of finite abelian groups, Transactions on Combinatorics, 3 no. 1 (2012) 5-10 (ISSN:2251-8665).*
5. *Gary Chartrand, Ping Zhang. Introduction to Graph Theory, Tata McGraw-Hill, 2006.*
6. *Joseph A. Gallian. Contemporary Abstract Algebra, Narosa Publishing House, 1999.*
7. *Sattanathan, M., and Kala, R., An Introduction to Order Prime Graph, Int. J. Contemp. Math. Sci., 4(10), 2009, 467-474.*
8. *] Sattanathan, M., and Kala, R., An Introduction to Order Prime Graph, Int. J. Contemp. Math. Sci., 4(10), 2009, 467-474.*
9. *T. Chalapathi, R.V.M.S.S. KiranKumar. Order divisor graphs of finite groups, Malaya journal of matematik, 2017, 5(2): 464-474.*