STAR CHROMATIC NUMBER AND STAR CHROMATIC INDEX OF SOME CYCLE RELATED GRAPHS

ABSTRACT

Graph Theory is a field of Mathematics that has a wide range of Applications in various aspects. Graph coloring, is one of the concepts of graph theory that can be incorporated into countless fields. Star coloring which was introduced by B. Grünbaum in 1973, is a type of path coloring of Graphs considering the vertices. Since, its introduction star coloring has been found for various graphs such that the concept was also extended to the edge version by Liu and Deng in 2008. In this paper, the star coloring and star edge coloring of some cycle related graphs and its respective chromatic number has been discussed. This includes the star chromatic number and star chromatic index for Tadpole graph, Pan graph and Friendship graph.

Keywords: *Graph coloring -Path coloring -Star coloring -Star edge coloring -Star chromatic number -Star Chromatic Index -Tadpole Graph -Pan Graph -Friendship Graph.*

1. Introduction:

Graph Coloring is a topic of Graph theory that was started in the year 1852 with the introduction of the four-color problem by Francis Guthrie. Later on, Graph coloring was developed into various kinds such as vertex coloring, edge coloring, total coloring, face coloring, etc., And when the coloring is restricted along the path, it is known as path coloring. Star coloring which was introduced by B. Grünbaum in 1973 is a type of path coloring in which we consider the vertices and its corresponding chromatic number was referred to as Star chromatic number. This concept was also extended to the edge version by Liu and Deng in the year 2008, this coloring was known as the Star edge coloring and its corresponding chromatic number was referred to as Star chromatic index. Since then, countless results and theorems have emerged under this topic. They have been applied to various graphs and several bounds have also been obtained. In the upcoming chapters, we have discussed the star chromatic number and star chromatic index for various cycle related graphs. This includes graphs such as Tadpole Graph, Pan Graph and Friendship Graph. These graphs have one or more cycles incorporated in their structure.

2. Preliminaries:

Definition 2.1[1]

A graph coloring is said to be a proper coloring if no two adjacent components (vertices (or) edges) are colored the same. In other words, the color used in adjacent components has to be distinct.

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Definition 2.2[1]

The Star vertex coloring introduced in 1973 by B. Grünbaum is said to be a proper coloring if no path of length 3 in the graph is bicolored. The minimum number of colors required to obtain a proper star coloring is known as Star Chromatic Number.

Definition 2.3[2]

The Star edge coloring introduced in 2008 by Liu and Deng is said to be a proper coloring if no path of length 4 in the graph is bicolored. The minimum number of colors required to obtain a proper star edge coloring is known as Star Chromatic Index.

In the following paragraphs, various graphs are defined for which the star chromatic number and star chromatic index is defined in the upcoming sections.

Definition 2.4[3]

The tadpole graph denoted as $T_{m,n}$ is defined as the graph consisting of a cycle graph C_m joined together with a path graph P_n by a common vertex. It is also known as Dragon Graph.

Definition 2.5[4]

The pan graph denoted as n-pan is defined as the graph consisting of a cycle graph C_n joined together with a Singleton graph K_1 by a common vertex. It has $n + 1$ vertices.

Definition 2.6[5]

The friendship graph denoted as $f_{m,n}$ is defined as the graph consisting of n copies of the cycle graph C_m joined together by a common vertex in the middle.

3.Star Vertex Coloring and Star Chromatic Number:

In this section, we have discussed the star coloring and star chromatic number for the graphs defined in the second section.

Theorem 3.1

The star vertex coloring of the tadpole graph is defined as

$$\chi_s(T_{m,n}) = \begin{cases} 4 & if\ m = 5 \\ 3 & otherwise \end{cases}$$

Proof:

The tadpole graph is found to be a combination of a cycle graph and path graph. The proof is given in two cases by considering the parameter m

Case 1: When $\boldsymbol{m \neq 5}$

As per the structure of the cycle graph and path graph, it is evident that a proper star coloring can be obtained using a minimum of 3 colors. Since, both the cycle and path graph can be star colored properly using a minimum of 3 colors. Assign the colors {1,2,3} to the vertices of the cycle graph by alternating between the three colors. Then assign a different color to the first vertex of the path graph than the color used in adjacent vertices of the cycle graph. Then by alternating the three colors, we can arrive at a proper star coloring. Thus $\chi_s(T_{m,n}) = 3$ when $m \neq 5$

Case 2: When $\boldsymbol{m = 5}$

Assign the colors {1,2,3} to the vertices of the cycle graph by alternating the colors, but this will not result in a proper coloring as at least one of the paths in the graph will become bicolored as per the assignment of colors. We need an addition of one more color so as no path will become bicolored. Then by assigning a different color to the first vertex of the path graph than the color used in adjacent vertices of the cycle graph.

And by alternating the three colors in the path graph, we get a proper star

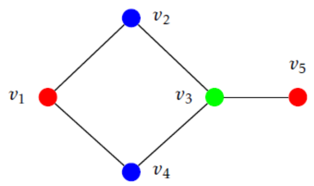


Figure 3.3

coloring. Thus $\chi_s(T_{m,n}) = 4$ when $m = 5$.

Example 3.2

Consider the tadpole graph $T_{6,4}$ having 10 vertices and 10 edges as seen in *Figure 3.1*. It is possible to color these vertices in such a way that no path is bicolored using a minimum of 3 colors. Thus, $\chi_s(T_{6,4}) = 3$

Example 3.3

Consider the tadpole graph $T_{5,4}$ having 9 vertices and 9 edges as seen in *Figure 3.2*. It is not possible to color these vertices in such a way that no path is bicolored using a

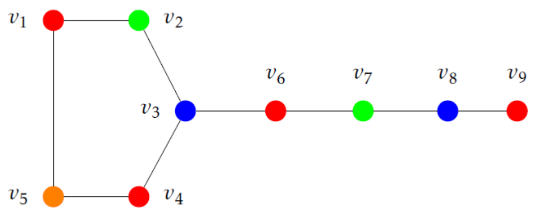


Figure 3.2

minimum of 3 colors. Therefore, an additional color is required to acquire a proper star coloring. Thus, $\chi_s(T_{6,4}) = 4$

Theorem 3.4

The star vertex coloring of the n −pan graph is defined as

$$\chi_s(n - pan) = \begin{cases} 4 & if\ n = 5 \\ 3 & otherwise \end{cases}$$

Proof:

The n −pan graph is isomorphic to the $(n, 1)$ −tadpole graph. The proof is similar to the previous theorem and can be got by substituting 1 for n

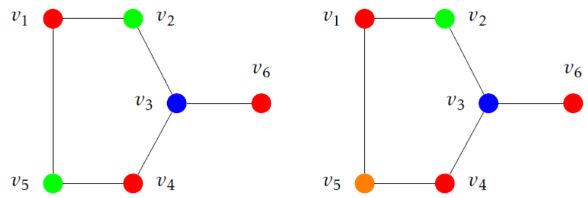


Figure 3.4 (a) and (b)

Example 3.5

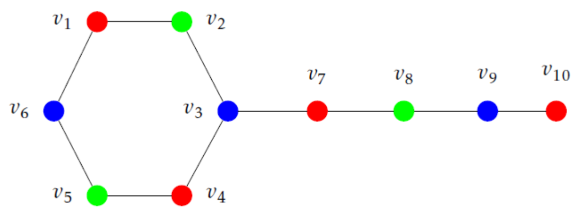


Figure 3.1

Consider the $4 - pan$ graph consisting of 5 vertices. We can color the graph using a minimum of 3 colors such that no path is bicolored. The 3-star coloring of $4 - pan$ can be seen in *Figure 3.3.* Therefore, $\chi_s(4 - pan) = 3$

Example 3.6

Consider the $5 - pan$ graph consisting of 6 vertices. It is evident from *Figure 3.4 (a)* that it is not possible to color the graph using a minimum of 3 colors as the path $\{v_2, v_1, v_5, v_4\}$ will become bicolored. Therefore, we need an additional color to make sure no path is bicolored as in *Figure 3.4 (b)*. Thus, $\chi_s(5 - pan) = 4$

Theorem 3.7

The star vertex coloring of the friendship graph is defined as

$$\chi_s(f_{m,n}) = \begin{cases} 4 & if\ m = 5 \\ 3 & otherwise \end{cases}$$

Proof:

As per the definition, friendship graph consists of n copies of C_m having a vertex as common, the proof is given by considering the number of vertices in the cycle

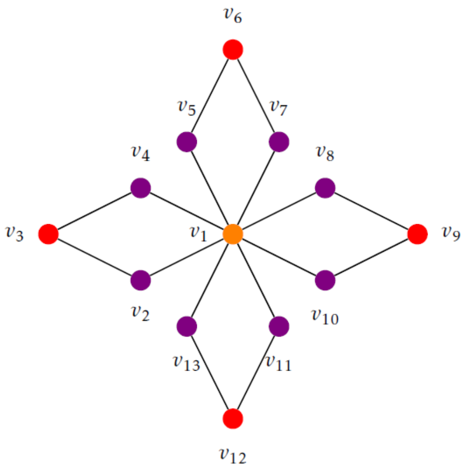


Figure 3.5

graph.

Case 1: When $\boldsymbol{m \neq 5}$

It is evident that we can obtain a proper star coloring for cycle graph using a minimum of 3 colors when $n \neq 5$. Assign 1 color to the middle vertex and assign colors from

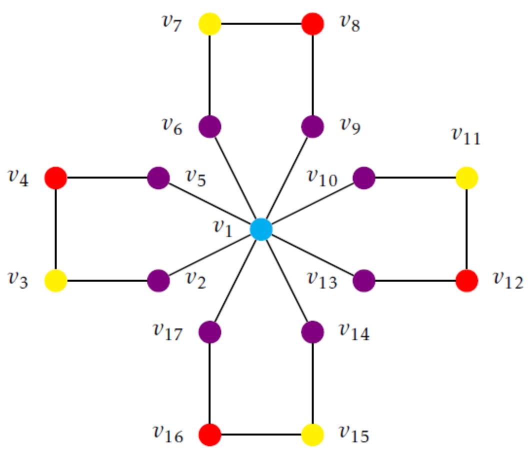


Figure 3.6

{1, 2, 3} for the remaining vertices. Thus, $\chi_s(f_{m,n}) = 3$ when $m \neq 5$

Case 2: When $\boldsymbol{m = 5}$

Assign 1 color to the middle vertex and assign colors {1, 2, 3} to the remaining vertices on both sides. But we will arrive at a contradiction on both sides as at least 2 paths will become bicolored as per this assignment. Therefore, we need another color 4 in order to avoid any path from becoming bicolored. Thus, $\chi_s(f_{m,n}) = 4$ when $m = 5$

Example 3.8

Consider the $f_{4,4}$ graph consisting of 13 vertices. By assigning color 1 to the common vertex, alternate between the remaining 2 colors to color the rest of the vertices. It is clear that no path is bicolored. Therefore, the coloring obtained is a proper one (refer *Figure 3.5*). Therefore, $\chi_s(f_{4,4}) = 3$.

Example 3.9

Consider the $f_{5,4}$ graph consisting of 17 vertices. By assigning color 1 to the common vertex, alternate between the remaining 3 colors to color the rest of the vertices (as it is not possible using 2 colors).

It is clear that no path is bicolored using 4 colors. Therefore, the coloring obtained is a proper one (refer *Figure 3.6*). Therefore, $\chi_s(f_{5,4}) = 4$.

4.Star Edge Coloring and Star Chromatic Index:

In this section, we have discussed the star edge coloring and star chromatic index for the graphs defined in the second section.

Theorem 4.1

The star edge coloring of the tadpole graph is defined as

$$\chi'_s(T_{m,n}) = \begin{cases} 4 & if\ m = 3,4,5 \\ 3 & otherwise \end{cases} \text{ and } n \geq 1$$

Proof:

The tadpole graph is found to be a combination of a cycle graph and path graph. The proof is given in four cases by considering the parameter m

Case 1: When $\boldsymbol{m \neq 3,4,5}$

Assign the colors {1,2,3} to the edges of the cycle graph by alternating between the three colors. Then assign a different color to the first edge of the path graph than the color used in adjacent edges of the cycle graph. Then by alternating the three colors, we can arrive at a proper star coloring such that no path is bicolored. Thus $\chi'_s(T_{m,n}) = 3$ when $m \neq 3,4,5$

Case 2: When $\boldsymbol{m = 3}$

Assign the colors {1,2,3} to the edges of the cycle graph by alternating the colors, then assign a different color to the first edge of the path graph than the color used in adjacent edges of the cycle graph. But this will not result in a proper coloring as at least one of the paths will become bicolored when the number of edges in the path graph increases. Therefore, an addition of one color is required to color the starting edge in the path graph. Then by alternating the color 4 and any two of the colors from {1, 2, 3} we will arrive at a proper star coloring. Thus, $\chi'_s(T_{m,n}) = 4$ when $m = 3$

Case 3: When $\boldsymbol{m = 4}$

Assign the colors {1,2,3} to the edges of the cycle graph by alternating the colors, then assign a different color to the first edge of the path graph than the color used in adjacent edges of the cycle graph. But this is not possible as at least one of the paths will become bicolored. Therefore, an additional color is required to color the starting edge in the path graph. Then by alternating the color 4 and any two colors from {1, 2, 3} we will arrive at a proper star coloring. Thus, $\chi'_s(T_{m,n}) = 4$ when $m = 4$

Case 4: When $\boldsymbol{m = 5}$

Assign the colors {1,2,3} to the edges of the cycle graph by alternating the colors, we will arrive at a contradiction as at least one of the paths will become bicolored. An addition of one more color is required to obtain a proper coloring of the cycle graph. Then by alternating between any three colors, we can get a proper coloring of the path. Thus, $\chi'_s(T_{m,n}) = 4$ when $m = 5$

Example 4.2

Consider the tadpole graph $T_{8,3}$ consisting of 11 vertices. It is possible to edge star color the graph properly by alternating between 3 colors. Therefore, $\chi'_s(T_{8,3}) = 3$ (refer *Figure 4.1)*

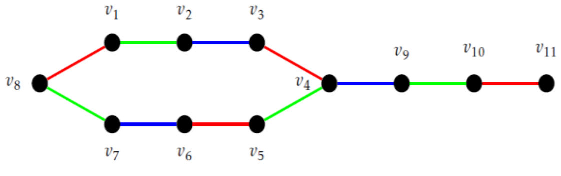
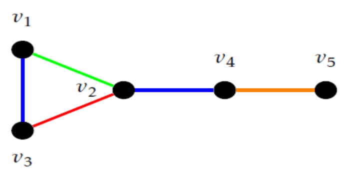


Figure 4.1

Example 4.3



Consider the tadpole graph $T_{3,2}$ consisting of 5 vertices. It is not possible using 3 colors as the path $v_3 - v_5$ will become bicolored. Thus, an additional color is required to ensure the proper coloring of the path. Therefore, $\chi'_s(T_{3,2}) = 4$ (refer *Figure 4.2*)

Example 4.4

Consider the tadpole graph $T_{4,3}$ consisting of 7 vertices. It is not possible using 3 colors as the path $v_2 - v_5$ will become bicolored. Thus, an additional color is required to ensure the proper coloring of the path joining the cycle. Therefore, $\chi'_s(T_{4,3}) = 4$ (refer *Figure 4.3*)

Example 4.5

Consider the tadpole graph $T_{5,3}$ consisting of 8 vertices. It is not possible using 3 colors as it requires at least 4 colors to properly color the cycle graph C_5. Therefore, an additional color is required to ensure the proper coloring of the graph. Thus, $\chi'_s(T_{5,3}) = 4$ (refer *Figure 4.4*)

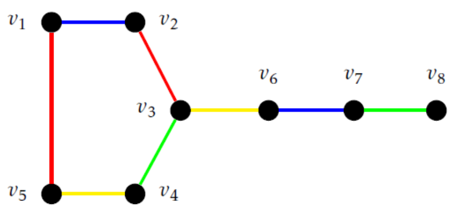


Figure 4.4

Theorem 4.6

The star edge coloring of the n –pan graph is defined as

$$\chi'_s(n - pan) = \begin{cases} 4 & if\ n = 4,5 \\ 3 & otherwise \end{cases}$$

Proof:

The n-pan graph is found to be a combination of a cycle graph and a singleton graph. The proof is given in three cases by considering the parameter n

Case 1: When $\boldsymbol{n \neq 4,5}$

Assign the colors {1,2,3} to the edges of the

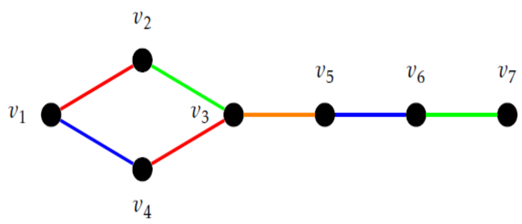


Figure 4.3

cycle graph by alternating between the three colors. Then assign a different color than the colors used in the adjacent edges of the cycle graph to the edge connecting the cycle and the singleton vertex. Thus, we can obtain a proper star coloring using a minimum of 3 colors. Thus, $\chi'_s(n - pan) = 3$ when $n \neq 4, 5$

Case 2: When $\boldsymbol{n = 4}$

Assign the colors {1,2,3} to the edges of the cycle graph by alternating between the three colors. Then the colors {2,3} will be assigned to the edges that are adjacent to the edge connecting the cycle graph to the singleton graph. Therefore, the color 1 is the only possible color to use. But we will arrive at a contradiction if we use the color 1, as at least one path connecting the cycle to the singleton graph will become bicolored. Thus, we require an additional color to obtain a proper star coloring. Thus, $\chi'_s(4 - pan) = 4$

Case 3: When $\boldsymbol{n = 5}$

We need a minimum of 4 colors to properly star color the cycle graph and the edge connecting the cycle and the singleton graph cam be connected using any one of the colors that is distinct than the one used in the adjacent edges of the cycle graph. Thus, $\chi_s'(5-pan)=4$

Example 4.7

Consider the $9-pan$ graph consisting of 10 vertices. It is possible to edge star color using a minimum of 3 colors by alternating between them. Thus, $\chi'_s(9-pan)=3$ (refer *Figure 4.5*)

Example 4.8

Consider the $4-pan$ graph consisting of 5 vertices. It is not possible to edge star color using a minimum of 3 as the path v_2-v_5 will become bicolored. And an additional color is required. Thus, $\chi'_s(4-pan)=4$ (refer *Figure 4.6*)

Example 4.9

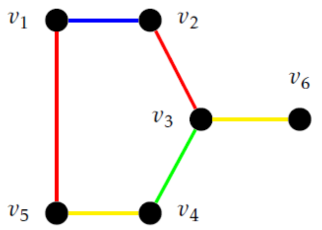


Figure 4.7

Consider the $5-pan$ graph consisting of 6 vertices. It is not possible to edge star color using a minimum of 3 as at least 4 colors are required to color the cycle properly. Therefore, it is evident from *Figure 4.7* that $\chi'_s(5-pan)=4$

Theorem 4.10

The star vertex coloring of the friendship graph is defined as

$$\chi'_s(f_{m,n})=2n \text{ where } n\geq 2$$

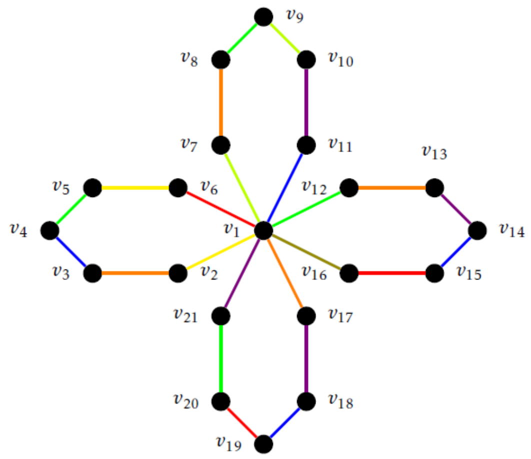


Figure 4.8

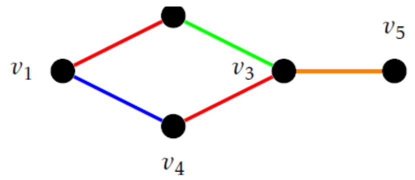


Figure 4.6

Proof:

As per its structure, the graph consists of n copies of cycle joined by a common vertex. The cycle graph can be star colored properly with at most 4 colors. But each vertex can be adjacent to at most two edges. The common vertex would be adjacent with n copies of the cycle graph and each would have 2 adjacent edges. Therefore, we need at least $2n$ colors to color the edges that are adjacent to the common vertex. Assign colors from $\{1,2,\ldots,2n\}$ to these edges. We can use the required colors form the same 2n colors to color the remaining edges such that no path is bicolored. Thus, we can conclude that $\chi'_s(f_{m,n})=2n$

Example 4.11

Consider the friendship graph $f_{6,4}$ having 21 vertices. As the middle vertex is adjacent to 8 edges. We require a minimum of 8 colors to properly color the graph. These 8 colors are more than enough to color the rest of the edges. Therefore $\chi'_s(f_{m,n}) = 2(4) = 8$, this can be seen in *Figure 4.8*.

5. Conclusion:

In this paper, we have found the star chromatic number and star chromatic index for various cycle related graphs such as Tadpole graph $T_{m,n}$, n −pan graph and friendship graph $f_{m,n}$ It is evident that it is possible to find the star chromatic number and star chromatic index for almost all graphs in existence. The Star coloring is found to be a very fascinating topic with astonishing applications in various fields including scheduling and sequencing.

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