TOTAL RESTRAINED DETOUR NUMBER OF FAMILY OF LADDER GRAPHS

ABSTRACT

In a connected graph $G = (V, E)$ of order $n \geq 2$, a set $S \subseteq V(G)$ is called a total restrained detour set in G if S is a detour set such that the subgraphs S and $V - S$ have no isolated vertices. The restrained detour number of G, denoted by $dn^{tr}(G)$, is the minimum cardinality of total restrained detour set of G. In this article, we introduce the total restrained detour number and determine the restrained detour number of standard graphs and the family of ladder graphs.

Keyword : *detour set, detour number, restrained detour set, restrained detour number.*

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies, we refer to [3]. For vertices u and v in a connected graph G, the detour distance $D(u, v)$ is the length of a longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ detour. A vertex x is said to lie on a $u - v$ detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V(G)$ is called a detour set if every vertex v in G lies on a detour joining a pair of vertices of S. The detour number $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a detour basis of G. These concepts were introduced by Chartrand et al. [2]. The detour concept was further studied by S. Athisayanathan et al. [1], [6], [8]. A total detour set of a graph G is a detour set S such that the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total detour set of G is the total detour number of G. Total detour number was extended to various concepts by number of authors in [7], [4]. Detour concept was extended to restrained detour concept in [5]. In this article, we introduce the total restrained detour number denoted by $dn^{tr}(G)$. We investigate the total restrained detour number for various ladder graphs.

2. Preliminaries

Definition 2.1: The Cartesian product G of two graphs G_1 and G_2, commonly denoted by $G_1 \square G_2$ or $G_1 \times G_2$, has vertex set $V(G) = V(G_1) \times V(G_2)$ and two distinct vertices (u, v) and (x, y) of $G_1 \square G_2$ are adjacent if either $(1) u = x \text{ and } vy \in E(G_2) \text{ or } (2) v = y \text{ and } ux \in E(G_1)$.

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Definition 2.2: The Cartesian product of P_n and K_2 is called ladder graph and is denoted as L_n, where $L_n = P_n \times K_2$. By removing the edges $v_1 u_1$ and $v_n u_n$ from L_n, an openladder denoted by $O(L_n)$.

Definition 2.3: A Triangular Ladder $TL_n, n \geq 2$ is a graph obtained from a ladder L_n by adding the edges $u_i v_{i+1}$ for $1 \leq i \leq n-1$, where u_i and v_i, $1 \leq i \leq n$ are the vertices of the two base paths of L_n. It is denoted as (TL_n). An Open Triangular Ladder is obtained from a triangle ladder by removing the edges $v_1 u_1$ and $v_n u_n$ and is denoted $O(TL_n)$.

Definition 2.4: A Diagonal Ladder graph DL_n, $n \geq 2$ is obtained from a ladder graph L_n by adding the edges $u_i v_{i+1}$ and $u_{i+1} v_i$ for $1 \leq i \leq n-1$ and denoted as DL_n. An Open Diagonal Ladder is obtained from a diagonal ladder graph by removing the edges and $u_i v_i$ and $u_n v_n$. It is denoted as $O(DL_n)$.

Definition 2.5: A Circular Ladder graph denoted by CL_n is obtained by joining parallel vertices of two cycles of same length.

Definition 2.6: A Mobius Ladder graph ML_n is a graph obtained from the ladder graph L_n by adding the edges $u_1 v_n$ and $v_1 u_n$. We denote it as ML_n for clarity though it its earlier denoted as M_n.

Definition 2.7: A Slanting Ladder SL_n is the graph obtained from two path length n say $P_n = (u_1, u_2, u_3, \dots \dots, u_n)$ and $P_n' = (v_1, v_2, v_3, \dots \dots, v_n)$ by joining u_{i+1} to v_i for $i = 1$ to $n-1$ to joining u_i to v_{i+1} for $i = 1$ to $n-1$.

Definition 2.8: Consider a Ladder graph on n vertices. Place it horizontally. Label the vertices as $v_{n1}, v_{n2}, v_{n3}, \dots, v_{nn}$ and $v_{(n-1)1}, v_{(n-1)2}, v_{(n-1)3}, \dots, v_{(n-1)n}$ such that $v_{11}, v_{12}, v_{13}, \dots, v_{1n}$ are the lowest path vertices. Attach a path on $n-1$ vertices parallely keeping the corresponding vertices. Next attach a path of length $n-2$ whose vertices are following a similar procedure. Repeat the process until a path of length 1 whose vertices are v_{11}, v_{12} is attached. The resulting graph looks as a stepping ladder and the graph is called Step Ladder and denoted as $S(T_n)$.

Definition 2.9: A set $S \subseteq V(G)$ is called a total restrained detour set in a connected graph G, if S is a detour set such that $G[S]$ and $G[V-S]$ have no isolated vertices. The restrained detour number of G, denoted by $dn^{tr}(G)$, is the minimum cardinality of total restrained detour set of G.

3. Total restrained detour number of the family of Ladder graphs

Theorem 3.1: For Ladder graph L_n, $dn^{tr}(L_n) = 2$.

Proof: Let $L_n = P_n \times K_2$ be a ladder of order $2n$ and of size $3n-2$. Let S be the detour set of L_n. If we consider the first and last vertices of the two copies of path P_n in $G = L_n$, then we have two cases.

Case (i): Let n be odd. Then either $S = \{u_1, v_n\}$ or $S = \{u_n, v_1\}$. Clearly, every vertex of L_n lie on the $u_1 - v_n$ or $u_n - v_1$ detour of length $2n-1$. Since $G[S]$ has isolated vertices, we consider $S' = \{u_l, v_l : l = 1 \text{ or } n\}$ that yields both $G[S']$ and $G[V-S']$ to have no isolated vertices. Thus S' is a minimum total restrained detour set and $dn^{tr}(G) = 2$.

Case (ii): Let n be even. Then either $S = \{u_1, u_n\}$ or $S = \{u_n, v_1\}$. Clearly, every vertex of L_n lie on the $u_1 - u_n$ or $v_1 -$

v_n detour of length $2n-1$. Here we find that $G[S]$ has isolated vertices. Therefore, we consider $S' = \{u_1, v_n\}$ or $\{u_n, v_1\}$. Thus, S' is a minimum total restrained detour set and so $dn^{tr}(G) = 2$. ■

Theorem 3.2: For an Open Ladder graph $O(L_n), dn^{tr}O(L_n) = 8$.

Proof: Let $G = O(L_n)$ be an open ladder with $|V(O(L_n))| = 2n$ and $|E(O(L_n))| = 3n-4$. Let $S = \{u_1, u_n, v_1, v_n\}$ be the set of all end-vertices of $O(L_n)$. Then we have two cases.

Case (i): Let n be even. Then every vertex of $O(L_n)$ lies on $u_1 - v_n$ detour of length $2n-3$. Thus $S = \{u_1, u_n, v_1, v_n\}$ is the unique minimum detour set such that the induced subgraph $G[V-S]$ has no isolated vertices.

Case (ii): Let n be odd. Then S is the detour set, for all the vertices of $O(L_n)$ lie on either $u_1 - v_n$ or $v_1 - v_m$ detour of length $2n-3$ and only $G[V-S]$ has no isolated vertices. Also the detours $u_1 - u_n$ and $v_1 - v_n$ of length $2(n-2)$ traverses all the vertices of $O(L_n)$. Here also $G[V-S]$ has no isolated vertices.

From the above cases we find that $G[S]$ has isolated vertices. Therefore, we consider $S' = S \cup S^a$, where S^a contains the adjacent vertices of the elements of S. Thus is S' a minimum total restrained detour set and $|S'| = 8$. Hence $dnt^r\big(O(L_n)\big) = 8$. ■

Theorem 3.3: For a Triangular Ladder $TL_n, dn^{tr}(TL_n) = 2$.

Proof: Let $G = TL_n$ be a triangular ladder with $|V(TL_n)| = 2n$ and $|E(TL_n)| = 4n-3$. Let $S = \{u_1, v_1\}$ or $S = \{u_n, v_n\}$ be a detour set. Then every vertex of TL_n lies on either $v_1 - u_m$ or $u_1 - v_1$ detour of length $2n-1$. Thus S is a minimum total restrained detour set of TL_n. Hence $dn^{tr}(TL_n) = 2$. ■

Theorem 3.4: For an Open Triangular ladder $O(TL_n)$, $dn^{tr}(O(TL_n)) = 4$.

Proof: Let $O(TL_n)$ be the open triangular ladder, where $|V(O(TL_n))| = 2n$ and $|E(O(TL_n))| = 4n-5$. Let $S' = \{v_1, u_n, v_2, u_{n-1}\}$ be the set of four vertices of $O(TL_n)$. Obviously, all the vertices of $O(TL_n)$ lie on the $v_1 - u_n$ detour of length $2n-1$. Since $G[S']$ and $G[V-S']$ has no isolated vertices S' is a minimum restrained detour set of $O(TL_n)$. Hence $dn^{tr}(O(TL_n)) = 4$. ■

Theorem 3.5: For a Diagonal Ladder $TL_n, dn^{tr}(DL_n) = 2$.

Proof: Let DL_n be a Diagonal ladder with $|V(DL_n)| = 2n$ and $|E(DL_n)| = 5n-4$. Then $S_1 = \{u_1, u_n\}$, $S_2 = \{v_1, v_n\}$, $S_3 = \{u_1, v_n\}$ and $S_4 = \{u_n, v_1\}$ be the detour sets. Then every vertex of DL_n lies on any detour of length $2n-1$ or of length $2(n-1)$. Since $< S_i >$ for $1 \le i \le 4$, have isolated vertices S_i is not a minimum total restrained detour set of DL_n. Consider the sets $S_5 = \{u_1, v_1\}$ and $S_6 = \{u_n, v_n\}$ that satisfy the condition of the definition with the detour $u_1 - v_1$. Hence, $dn^{tr}(DL_n) = 2$. ■

Theorem 3.6: For an Open Diagonal Ladder $O(DL_n)$, $dn^{tr}(O(DL_n)) = 3$.

Proof: Let $O(DL_n)$ be the open diagonal ladder, where $|V(O(DL_n))| = 2n$ and $|E(O(DL_n))| = 5n-6$. Let $S = \{u_k, v_k\}$ be the set of first vertices or the n^{th} vertices of $O(DL_n)$. Obviously, all the vertices of $O(DL_n)$ lie on the $u_k - v_k$ detours of length $2(n-1)$. Since $G[S]$ has isolated vertices, S is not a minimum total restrained detour set of $O(DL_n)$. Let $S' = S \cup \{v\}$, where $v \in \{u_{k\pm1}, v_{k\pm1}\}$. Hence $dn^{tr}(G) = 3$. ■

Theorem 3.7: For a Circular Ladder graph $CL_{n,}\ dn^{tr}(CL_n) = 2$.

Proof: Let $G = CL_n$ be a circular ladder with $|V(CL_n)| = 2n$ and $|E(CL_n)| = 3n$. Consider a set S of two vertices of CL_n.

Case (i): Suppose $S = \{u_i, v_i: i = 1\ or\ n\}$. Then every vertex of CL_n lies on $u_i - v_i$ detour of length $2n-1$. Thus S is a minimum total restrained detour set. Since $G[S]$ and $G[V-S]$ have no isolated vertices, $dn^{tr}(CL_n) = 2$.

Case (ii): Consider $S = \{u_i, u_{i+1}: i = 1\ or\ n-1\}$.Consider $S = \{v_j, v_{j-1}: j = 2\ or\ n\}$. Then all the verities of CL_n lies on $u_i - u_{i+1}$ detour of length $2n-1$. Clearly, $G[S]$ and $G[V-S]$ are without isolated vertices and so S is a minimum total restrained detour set of $G = CL_n$. Hence, $dn^{tr}(G) = 2$.

Case (iii): Suppose $S = \{v_k, u_l: k, l = 1 or\ n; k \neq\ l\ \}$. Then every vertex of CL_n lies on some $v_k - u_l$ detour. Obviously, S is the minimum total restrained detour set for the induced subgraphs $G[S]$ and $G[V(CL_n) - S]$ have no isolated vertices. Thus $dn^{tr}(G) = 2$. ■

Theorem 3.8: For a Mobius Ladder graph $ML_{n,}\ dn^{tr}(ML_n) = 2$.

Proof: Let $G = ML_n$ be a Mobius ladder with $|V(ML_n)| = 2n$ and $|E(ML_n)| = 3n$. Let $S_1 = \{u_1, v_1\}$ and $S_2 = \{u_n, v_n\}$ be the minimum total restrained detour sets. Then every vertex of ML_n lies on $u_1 - v_1$ detour or $u_n - v_n$ detour of length $2n-1$. Thus $dn^{tr}(G) = 2$. ■

Theorem 3.9: For a Slanting Ladder graph $SL_{n,}\ dn^{tr}(SL_n) = 4$.

Proof: Let $G = SL_n$ be a slanting ladder with $|V(SL_n)| = 2n$ and $|E(SL_n)| = 3(n-1)$. Consider a set $S = \{u_1, u_2, v_{n-1}, v_n\}$ of two end-vertices and their support vertices. Then every vertex of SL_n lies on $u_1 - v_n$ detours of length $n+1$. Since $G[S]$ and $G[V-S]$ have no isolated vertices, it follows that S is a minimum total restrained detour set. Hence $dn^{tr}(G) = 4$. ■

Theorem 3.10: For a Step Ladder graph $ST_{n,}\ dn^{tr}(ST_n) = 2$.

Proof: Let $G = ST_n$ be a step ladder with $|V(ST_n)| = \frac{n^2+3n-2}{2}$ and $|E(ST_n)| = n^2 + n - 2$. Consider a set $S = \{v_{11}, v_{12}\}$ of two top vertices of ST_n. Then every vertex of SL_n lies on $v_{11} - v_{12}$ detours. The first detour path P_1 of length $\frac{n^2+3(n-4)}{2}$ includes all the vertices of ST_n, except the vertices $v_{(2k+1)2(k+1)}$ for $k = 1,2,\ldots,\left\lfloor\frac{n}{2} - 1\right\rfloor$ when n is odd and except the vertices $v_{2k(2k+1)}$ for $k = 1,2,\ldots,\frac{n-2}{2}$ when n is even. However, these vertices are traversed by the second detour path P_2 of same length. Thus $G[S]$ and $G[V-S]$ have no isolated vertices. Therefore, $dn^{tr}(ST_n) = 2$. ■

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